ELEMENTS

OF THE

DIFFERENTIAL AND INTEGRAL CALCULUS

With Examples and Practical Applications

BY

J. W. NICHOLSON, A.M., LL.D.

President and Professor of Mathematics
Louisiana State University and Agricultural and Mechanical College

NEW YORK AND NEW ORLEANS
UNIVERSITY PUBLISHING COMPANY
1896

*** 1673

PREFACE.

In many respects this work is quite different from any other on the same subject, though in preparing it there has been no attempt at originality beyond presenting the principles in a more tangible form than usual, and thus securing a better text-book for the ordinary student of mathematics. The aim has been to prepare a work for beginners, and at the same time to make it sufficiently comprehensive for the requirements of the usual undergraduate course.

The chief distinction of the treatise is that it is based on the conception of *Proportional Variations.* This method has been employed as the most elementary and practical, and none the less rigorous or general, form of presenting the first principles of the subject (see the following Note).

Differentiation and Integration are carried on together, and the early introduction of practical applications both of the differential and integral calculus, which this mode of presenting the subject permits, is intended to serve an important purpose in illustrating the utility and potentiality of the science, and arousing the interest of the student. The formulas for differentiating and integrating are, as a rule, expressed in terms of u and v instead of x, u and v being functions of x. The advantages thus secured are obvious.

Among the additional features of special interest may be mentioned the following : (1) The treatment of dx as a variable independent of x (Art. 68, and Appendix, A_3); (2) a rigorous deduction of a simple test of absolute convergency, without recourse to the remainder in Taylor's formula (Arts. 115 to 119); (3) an extension of the ordinary rules for finding maxima and minima (Arts. 140 to 143); (4) a chapter on Independent Integration (Chap. IX); (5) integration by indeterminate co-

efficients (Arts. 211 to 216); (6) the introduction of *turns* in curve-tracing (Arts. 175 to 179); and (7) a new proof of Taylor's formula, which is believed to be as rigorous as, and less artificial than, those in general use (Appendix, A_5).

In preparing the book the best available authors have been consulted, and many of the examples have been taken from the works of Todhunter, Williamson, Courtenay, Byerly, Rice and Johnson, Taylor, Osborne, Loomis, and Bowser.

I improve this opportunity to tender my thanks to Prof. William Hoover of the Ohio University, Prof. Alfred Hume of the University of Mississippi, and Prof. O. D. Smith of the Polytechnic Institute of Alabama for valuable assistance in the reading of proofs. Their corrections and suggestions have relieved the treatise from various imperfections it would otherwise have contained.

Further acknowledgments of indebtedness are also due to my colleague, Prof. C. Alphonso Smith, of the Department of English, who has aided me with his scholarly criticisms.

JAMES W. NICHOLSON.

BATON ROUGE, LA., 1896.

NOTE.—The method of Proportional Variations, which is the suggestion and outgrowth of work in the class-room, is believed to possess the following merits:

(*a*) The conception is one with which the student is already familiar, for the principle of proportional changes is among the first that he encounters, even in the lower mathematics.

(*b*) It affords finite differentials, and, without introducing infinitesimals, or infinitely small quantities, or "the foreign element of time," has all the advantages of the differential notation.

(*c*) In many cases the proportional variations (or differential) can be detected by inspection (see Arts. 31, 32, 35), and in all cases they may be deduced by the theory of limits. Hence the method has all the lucidity of finite differences and all the rigor of the doctrine of limits.

(*d*) It is a method to which the doctrines of Infinitesimals and Rate of Change are easy corollaries.

(*e*) In general, the form and properties of the increments of all quantities are due to proportion and acceleration, or to proportional and disproportional changes; hence, a system of Calculus based on such changes adapts itself naturally to questions in Geometry, Mechanics, and Physics.

CONTENTS.

CHAPTER I.

FUNDAMENTAL PRINCIPLES.

[Pages 1 to 20.]

CHAPTER II.

ELEMENTARY DIFFERENTIATION AND INTEGRATION.

[Pages 21 to 42.]

---

* Similarly, the following numbers refer to articles and not to pages.

CHAPTER III.

SUCCESSIVE DIFFERENTIALS AND RATE OF CHANGE.

CHAPTER IV.

GENERAL DIFFERENTIATION.

CHAPTER V.

SERIES, DEVELOPMENT OF FUNCTIONS, AND INDETERMINATE FORMS.

CHAPTER VI.

MAXIMA AND MINIMA.

CHAPTER VII.

APPLICATIONS OF THE DIFFERENTIAL CALCULUS TO PLANE CURVES.

CHAPTER VIII.

GENERAL DEPENDENT INTEGRATION.

[Pages 160 to 190.]

CHAPTER IX.

INTEGRATION CONTINUED.

[Pages 191 to 210.]

CHAPTER X.

INTEGRATION AS A SUMMATION OF ELEMENTS.

APPENDIX.

LIMITED COURSES.

(*a*) **The first three chapters.** This course is complete as far as it goes, since differentiation and integration are carried on together. It embraces the notation, fundamental principles, and some of the most important applications of the Calculus. The student who understands elementary algebra and geometry, and the construction of elementary loci, should find but little difficulty in mastering it.

(*b*) **The first six chapters.** This adds to the former course the transcendental functions, development of functions, evaluation of the indeterminate forms, and maxima and minima.

Suggestions. (1) It is recommended to omit the more difficult examples and problems in passing over the book the first time.

(2) A_5 of the Appendix may be substituted for Arts. 116 to 122, at the discretion of the teacher.

DIFFERENTIAL AND INTEGRAL CALCULUS.

CHAPTER I.

FUNDAMENTAL PRINCIPLES.

QUANTITY.

1. THERE are two kinds of quantities employed in Calculus, *variables* and *constants.*

Variables are quantities whose values are to be considered as changing or changeable. They are usually represented by the final letters of the alphabet.

Constants are quantities whose values are not to be considered as changing or changeable. They are usually represented by the first letters of the alphabet. Particular values of variables are constants.

2. Dependent and Independent Variables. A Dependent Variable is one that depends upon another variable for its value, and an Independent Variable is one that does not depend on another variable, but one to which any arbitrary value or change of value may be assigned. In the elementary differential and integral calculus, the independent variable is usually restricted to real values.

Thus, in $u = x^2 - 7x + 5$, $v = (1 - x^2)^3$, $y = \log (1 + x)$, u, v and y are dependent variables, since they depend on the

variable x for their values; but x is an independent variable, since, as we may suppose, any value may be assigned to it without reference to any other variable.

3. Functions. Dependent variables are usually called functions of the variables on which they depend. Hence, one variable is a function of another when the first depends upon the second for its value, or when the two are so related that changes in the value of the latter produce changes in the value of the former.

Thus, the *area* of a varying square is a function of its *side*; the *cost* of cloth is a function of the *quality* and *quantity*; the *space* described by a falling body is a function of the *time*; every mathematical expression depending on x for its value, as x^2, $(3x - 7)^3$, $5x^2 - 6x + 11$, etc., is a function of x.

4. Increasing and Decreasing Functions. An Increasing Function is one that increases when the variable increases, as $(x+1)^2$, $3x^3$, $\log (5+x)$; and a Decreasing Function is one that decreases when the variable increases, as $\sqrt{10 - x^2}$, $\frac{5}{x}$, etc.

A function of x may be increasing for certain values of x, and decreasing for other values.

Thus, $y = x^2 - 4x + 5$ is a decreasing function for all values of $x < 2$, but increasing for all values of $x > 2$.

5. Explicit and Implicit Functions. An Explicit Function is one whose value is directly expressed in terms of the variable and constants.

Thus, in the equations $y = (a - x)^3$, $y = x^2 + 3x + 5$, y is an explicit function of x.

An Implicit Function is one whose value is implied in an equation, but not expressed directly in terms of the variable and constants.

Thus, in the equation $x^2 + 2xy + 5y = 10$, y is an implicit function of x, and x is an implicit function of y. By solving the equation for x or y, the function becomes explicit.

6. Algebraic and Transcendental Functions. One variable is called an Algebraic Function of another when the two are connected by an algebraic equation; that is, an equation which contains a finite number of terms involving only constant integral powers of the variables, or an equation which admits of being reduced to this form.

Thus, in $y = x^2 - 5x$, or $x^2y^2 - xy^5 + 8xy - 5 = 0$, or $y^{\frac{3}{4}} - \sqrt[3]{ax^2 + y} = 7$, y is an algebraic function of x, and *vice versa*.

If two variables are connected by an equation which is not algebraic, each is called a Transcendental Function of the other.

Thus, if $y = \sin x$, y is a transcendental function of x, and x of y.

The following are the elementary transcendental functions:

A **Logarithmic Function** is one that involves the logarithm of a variable; as, $\log x$, $\log (a + y)$.

An **Exponential Function** is one in which the variable enters as an exponent; as, a^x, y^x.

A **Trigonometric Function** is the sine, cosine, tangent, etc., of a variable angle; as, $\sin x$, $\cos y$.

An **Inverse-Trigonometric Function** is an angle whose sine, cosine, tangent, etc., is a variable; as, $\sin^{-1} x$, $\cos^{-1} y$, $\tan^{-1} t$, etc., which are read, "an angle whose sine is x," "an angle whose cosine is y," etc.

7. Continuous Functions. A function of a variable is continuous between certain values of the variable (1) when it has a finite value for every value of the variable, and (2) when the changes in its value corresponding to indefinitely small changes in the value of the variable are themselves indefinitely small.

Thus, in $y = ax + b$, or $y = \sin x$, or $y = e^x$, y is continuous for all finite real values of x; so also in $y = \sqrt{a^2 - x^2}$, but as a real quantity only for real values of $x > -a$ and $< a$. Again, $y = \dfrac{x}{x - 2}$ is not continuous between the limits $x = 1$ and $x = 3$, for when $x = 2$, $y = \infty$.

8. Notation of Functions. The symbol $f(x)$ is used to denote any function of x, and is read, "function of x." To denote different functions of x we employ other symbols, as $F(x)$, $f'(x)$, $\phi(x)$, $\theta(x)$, etc. According to this notation, $y = f(x)$ represents any equation between x and y when solved for y.

Thus, solving the equation $y^2 - 2axy + bx^3 = 0$ for y, we obtain $y = ax \pm \sqrt{a^2x^2 - bx^3}$, or $y = f(x)$.

The result of substituting any number, as m, for x in $f(x)$ is denoted by $f(m)$.

Thus, if $f(x) = x^2 - 5x + 6$,

$$\begin{aligned} f(0) &= 0^2 - 5 \cdot 0 + 6 = 6, \\ f(1) &= 1^2 - 5 \cdot 1 + 6 = 2, \\ f(2) &= 2^2 - 5 \cdot 2 + 6 = 0, \\ f(3) &= 3^2 - 5 \cdot 3 + 6 = 0, \\ f(4) &= 4^2 - 5 \cdot 4 + 6 = 2, \\ &\text{etc.,} \qquad \text{etc.} \end{aligned}$$

In $f(x)$ if x be increased by h the result is denoted by $f(x + h)$.

Thus, if $f(x) = x^2 + 5x$, then

$$\begin{aligned} f(x + h) &= (x + h)^2 + 5(x + h) \\ &= x^2 + 5x + (2x + 5)h + h^2. \end{aligned}$$

EXAMPLES.

1. In the function $f(x) = x^2 - 9x + 14$, (1) which are the constants? (2) Which is the variable? (3) Find the values of $f(0), f(2), f(7)$. (4) Which is the least: $f(3), f(5)$ or $f(6)$?

Ans. (1) 9 and 14; (2) x; (3) 14, 0, 0; (4) $f(5)$.

2. Given $f(x) = x^2 - 10x + 24$; (1) show that $f(3) - f(7) = 0$; (2) that $f(5) < f(4)$; (3) that $f(-1) = f(11)$; (4) that $f(x + h) = x^2 - 10x + 24 + (2x - 10)h + h^2$.

3. Reduce $2x + 3y + 12 = 0$ to the form $y = f(x)$.

$$y = -\tfrac{1}{3}(12 + 2x).$$

4. Reduce $x^2 + y^2 = R^2$ to the form $y = f(x)$.

$$y = \pm\sqrt{R^2 - x^2}.$$

5. Given $f(x) = -4 - \frac{2}{3}x$; show that $f(0) - f(3) = 2$.

6. Given $f(x) = \sqrt{4mx}$; show that $f(4m) - f(m) = 2m$.

7. If $f(x) = \sqrt{100 - x^2}$, show that $f(6) = f(8) + 2$.

8. In $a^2y^2 + b^2x^2 = a^2b^2$, is y a function of x? Why? What function?

(1) It is. (2) Because any change in the value of x produces a change in the value of y. (3) $\pm\frac{b}{a}\sqrt{a^2 - x^2}$.

INCREMENTS.

9. If the independent variable be made to change from one value to another, the quantity by which it is changed is called its **Increment**; and this increment is positive or negative according as the variable is increasing or decreasing.

When the independent variable receives an increment, the corresponding change in the value of any function of it is the increment of this function, and is found by subtracting the old from the new value of the function. Hence, if the function is increasing, its increment is positive; and if decreasing, its increment is negative.

The increment of a variable is denoted by writing the letter Δ (*delta*) before it. Thus, Δx does not mean Δ times x, but "the increment of x," and is so read. Similarly, Δu, $\Delta(x^3)$, and $\Delta(x^2 + 7x)$ denote the increments of u, x^3 and $x^2 + 7x$.

In this book h will generally be used instead of Δx, as it is more convenient; but it should always be remembered that $h = \Delta x$.

Increments and the method of finding them are illustrated algebraically and graphically in the solution of the following examples.

Illustrations. 1. If x is increased by h, what will be the increment of the function $u = cx$?

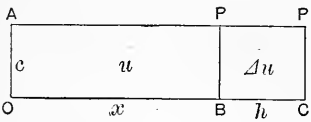

FIG. 1.

The value of u $(= cx)$ may be represented graphically by the area of the rectangle $OBPA$, whose base $OB = x$, and whose altitude $OA = c$.

$$\therefore \qquad u = cx = OBPA. \quad \ldots \ldots \quad (1)$$

When x is increased by $BC\,(= h)$, the new value of u will be $c(x + h)$, or area of $OCP'A$. Hence, denoting the increment of u by Δu, we have

$$u + \Delta u = cx + ch = OCP'A. \quad \ldots \ldots \quad (2)$$

Subtracting (1) from (2), member from member, we have

$$\Delta u = ch = BCP'P. \quad \ldots \ldots \quad (3)$$

That is, the increment of cx is c times the increment of x.

2. By how much will $u = x^2$ be increased when x is increased by h?

The value of u $(= x^2)$ may be represented graphically by the area of the triangle OBP whose base $OB = x$ and whose altitude $BP = 2x$. Hence

$$u = x^2 = OBP. \quad \ldots \quad (1)$$

When OB $(= x)$ is increased by $BC\,(= h)$ the new values of u, x^2 and OBP will be, respectively, $u + \Delta u$, $(x + h)^2$ and OCP', thus changing (1) into

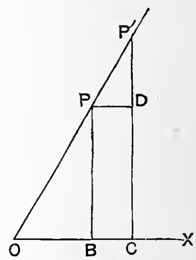

FIG. 2.

$$u + \Delta u = x^2 + 2xh + h^2 = OCP'. \quad \ldots \quad (2)$$

Subtracting (1) from (2), member from member, we have

$$\Delta u = 2xh + h^2 = BCP'P;$$

or

$$\Delta u = 2xh + h^2 = BP \times h + PDP'. \quad \ldots \quad (3)$$

3. Find the increment of

$$y = x^2 - 4x + 5. \qquad (1)$$

We may regard $y = x^2 - 4x + 5$ as the equation of a curve APP'; then, P being any point of the curve, $x = OB$ and $y = BP$.

When OB $(= x)$ is increased by $BC\,(= h)$, the new values of y, x^2-4x+5 and BP will be, respectively, $y + \Delta y$, $(x+h)^2 - 4(x+h) + 5$, and CP', thus changing (1) into

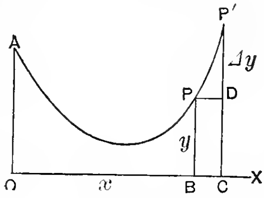

FIG. 3.

$$y + \Delta y = x^2 + 2xh + h^2 - 4x - 4h + 5 = CP'. \qquad (2)$$

Subtracting (1) from (2), we have

$$\Delta y = (2x - 4)h + h^2 = CP' - BP = DP'. \qquad (3)$$

COR. I. Evidently DP' will be $+$ or $-$ according as CP' is greater or less than BP; that is, in general, Δy will be positive or negative according as y is increasing or decreasing.

10. General Formula. To find the increment of

$$u = f(x). \qquad (1)$$

Increasing x by h, and denoting the corresponding increment of u by Δu, we have

$$u + \Delta u = f(x + h). \qquad (2)$$

$$(2) - (1), \qquad \Delta u = f(x + h) - f(x).$$

EXAMPLES.

1. Find the increment of $u = \pi x^2$, or the area of a concentric ring whose width is h and whose inner radius is x.

$$\Delta u = \pi(2x + h)h.$$

2. Find the increment of the cube $u = x^3$.

$$\Delta u = 3x^2h + 3xh^2 + h^3.$$

3. Find the increment of $f(x) = x^3 - 7x + 9$.

$$f(x+h) - f(x) = (3x^2 - 7)h + 3xh^2 + h^3.$$

4. Given $f(x) = x^3 + 2x^2 + 9$; show that

$$f(x+h) - f(x) = (3x^2 + 4x)h + (3x+2)h^2 + h^3.$$

5. Given $f(x) = \sqrt{x}$; prove that

$$f(x+h) - f(x) = \frac{h}{\sqrt{x+h} + \sqrt{x}}.$$

6. Given $u = \frac{1}{x}$; prove that $\Delta u = -\frac{h}{x^2 + xh}$.

VARIATION.

11. Proportional Variation. One quantity is said to vary proportionally with, or to vary as, another when the ratio of the one to the other remains constant.

The sign of variation is $\propto$. Thus, y varies as x is written $y \propto x$.

Illustrations. 1. The cost per yard of cloth remaining the same, the entire cost (y) varies as the quantity (x). That is,

$$y \propto x. \qquad (1)$$

2. The space (s) described by a body moving with a uniform velocity (v) varies as the time (t). Or

$$s \propto t. \qquad (2)$$

3. The area (u) of a rectangle having a constant altitude (a) varies as the base (x). Or

$$u \propto x. \qquad (3)$$

12. Principles. I. *If one quantity varies as another, one of them is a constant multiple of the other.*

Let $\qquad y \propto x,$

then $\qquad \frac{y}{x} = m$, a constant;

hence, $\qquad y = mx.$

COR. I. The variations (1), (2) and (3), Art. 11, may be written, respectively,

$y = mx$, where m is the price per yard of cloth;
$s = mt$, where m is the velocity of the body;
$u = mx$, where m is the altitude of the rectangle.

13. II. *If one variable is equal to a constant multiple of another, the first varies as the second.*

Let $$y = mx,$$
where m is a constant. Then
$$\frac{y}{x} = m, \quad \text{or} \quad y \propto x.$$

14. III. *If each of two quantities (u and v) varies as a third (x), the first quantity (u) varies as the second (v).*

Since $u \propto x$, $$u = mx; \qquad (1)$$
and since $v \propto x$, $$v = nx. \qquad (2)$$

(1) ÷ (2), $$\frac{u}{v} = \frac{m}{n}, \quad \text{or} \quad u = \frac{m}{n}v.$$

That is, $$u \propto v.$$

15. IV. *The product of two variables does not vary proportionally with either of the variables.*

Let us suppose $$yx \propto x;$$
then $$yx = mx, \quad \text{or} \quad y = m,$$
which is contrary to the hypothesis, since m is a constant.

16. Disproportional Variation. When one quantity does not vary as another, the variation is said to be disproportional. Hence, one quantity varies disproportionally with another when the ratio of the one to the other does not remain constant.

Thus, x^2 varies disproportionally with x, since $x^2 \div x \ (= x)$ is not a constant.

17. Principles. I. *The product of two variables varies disproportionally with either of the variables* (Art. 15).

18. II. *The nth power of a variable, n having any value except $+1$, varies disproportionally with the variable.*

For, $x^n \div x$ $(= x^{n-1})$ is a constant only when $n = +1$.

19. *If $m_2 h$ vanishes with h, $m_2 h^2$ varies disproportionally with h.*

Let us suppose $$m_2 h^2 \propto h;$$

then $$m_2 h^2 = mh, \quad \text{or} \quad m_2 h = m;$$

this being true for all values of h, must be true when $h = 0$, hence, since $m_2 h$ vanishes with h, the value of the constant m is 0. Therefore $m_2 h^2$ varies disproportionally with h if m_2 is of such a character that $m_2 h$ vanishes with h.*

20. Composition of Increments. In general the increment of any function of a single variable is composed of two parts, one of which changes proportionally, and the other disproportionally, with the increment of the variable.

Thus, let $y = f(x)$ represent any function of x, h any variable increment of x estimated from any particular value of x, and $\Delta y (= f(x+h) - f(x))$ the corresponding increment of y; then

$$\Delta y = m_1 h + m_2 h^2, \quad \ldots \ldots \quad (1)$$

where m_1 is constant with respect to h, and m_2 is of such a character that $m_2 h$ vanishes with h (Art. 19).

The proof of this property of increments will be given in

---

* To say that $m_2 h$ vanishes with h is equivalent to saying that m_2 does not involve any negative power of h, such as $\frac{a}{h}$, $\frac{b}{h^2}$, etc.; for if it did, the value of $m_2 h$, when $h = 0$, would be finite or infinite. In general, $m_2 h^2$ varies disproportionally with h whatever may be the character of m_2, but the only cases we shall have occasion to consider are those in which $m_2 h$ vanishes with h.

Art. 24; we give here two examples in illustration of this important proposition.

1. Take the increment of $u = x^2$, which we have found to be $\Delta u = 2xh + h^2$.

By reference to Fig. 2, it will be seen that $BCDP = 2xh$ and $PDP' = h^2$. Regarding BP $(= 2x)$ as constant, and BC $(= h)$ as variable, the first part of Δu, $BCDP$, varies proportionally with h, and the second part, PDP', disproportionally with h.

By comparison with (1), $m_1 = 2x$ and $m_2 = 1$.

2. Take the function $u = x^3 - 7x + 9$, the increment of which we have found to be

$$\Delta u = (3x^2 - 7)h + (3x + h)h^2.$$

Here, regarding x as constant, the part $(3x^2 - 7)h$ varies as h (Art. 13), and the other part, $(3x + h)h^2$, changes disproportionally with h (Art. 19). Comparing this increment with (1), we have

$$m_1 = 3x^2 - 7 \quad \text{and} \quad m_2 = 3x + h.$$

THEORY OF LIMITS.

21. The principles of limits, in addition to other merits, afford an admirable method and means of finding the proportional increments of related variables. For convenience of reference, and in order that the method of proportional changes and that of limits may be made to throw light upon each other, we give here a statement of such of the principles of limits as we shall have occasion to employ.

The **Limit** of a variable is a constant which the variable approaches, and from which it can be made to differ by less than any quantity which may be assigned, but which, on the other hand, it can never actually reach.

Thus, if the number of sides of a regular polygon inscribed in, or circumscribed about, a circle be indefinitely increased, the area of the circle will be the limit of the area of either polygon, and the circumference will be the limit of the perimeter of either.

Again, let $s = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \text{etc.}$

By increasing the number of terms of this series, the value of s approaches 2 indefinitely, but can never reach it; therefore 2 is the limit of s.

22. Principles. I. *The difference between a variable and its limit is a variable whose limit is* 0.

23. II. *If two variables are continually equal, and each approaches a limit, their limits are equal.*

For, let $X = Y$ be the variables and A and B their respective limits, and suppose $X + x = A$ and $Y + y = B$, then $x - y = A - B$; but the limits of x and y are 0 (Art. 22), therefore, at the limit, $A - B = 0$, or $A = B$.

24. Proof of the Formula $\Delta y = m_1 h + m_2 h^2$. . (Art. 20)

Let $y = f(x)$ be a continuous function such that

$$\frac{\Delta y}{\Delta x} \text{ or } \frac{f(x+h) - f(x)}{h}$$

approaches a definite limit (say m_1) as h approaches zero,* then the value of $\frac{\Delta y}{\Delta x}$ must be of the form $m_1 + m_2 h$, where $m_2 h$ is a quantity whose limit is 0 or one which vanishes with h.

That is, $\frac{\Delta y}{\Delta x} = m_1 + m_2 h^2$. $\therefore \Delta y = m_1 h + m_2 h^2$,

where m_1 is evidently independent of h.

Geometrical Illustration.† Let $y = f(x)$, as defined above, be the equation of a curve APm, where $x = OB$ and $y = BP$. Let TPt be a tangent to the curve at P. When x is increased by $BC = h$, we have $\Delta y = DP'$. Draw the secant SPP', and we have

$$\frac{\Delta y}{\Delta x} = \frac{DP'}{PD} = \tan XSP. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

---

* See Appendix, A_1. † For another illustration see Appendix, A_2.

Now taking the limits of the two members of (1), remembering that as h approaches 0, P' approaches P, and XSP approaches XTP as its limit, we have

$$m_1 = \tan XTP,$$

which is a definite quantity dependent on a particular value ($x' = OB$) of x, and independent of h.

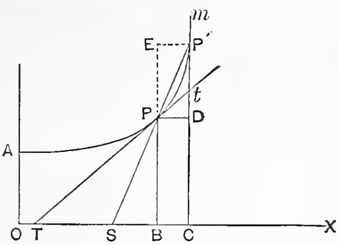

FIG. 4.

Again, since $m_1 = \tan XTP$, $m_1h = Dt$, and since

$$\Delta y = DP' = m_1h + m_2h^2,$$

$m_2h^2 = tP'$, which is a quantity that vanishes with h.

DIFFERENTIALS AND ACCELERATIONS.

25. The **Differential** of a function is that part of its increment which varies proportionally with the increment of the independent variable, and the **Acceleration** is that part which varies disproportionally with the increment of that variable.

Thus, Art. 20, (1), the differential of y is m_1h, and the acceleration of y is m_2h^2.

The differential of a quantity is denoted by writing the letter d before it. Thus, dy is not $d \times y$, but the differential of y, and is so read. The differentials of functions like x^3, $x^2 + 7x$, and $\sqrt{1 + x^2}$, are denoted by $d(x^3)$, $d(x^2 + 7x)$, and $d(\sqrt{1 + x^2})$.

Similarly, the acceleration of a function will often be denoted by writing the letter a before it; as, ay, which is read, "the acceleration of y."

COR. I. The increment of a function is equal to the sum of its differential and its acceleration. Thus,

$$\Delta y = dy + ay.$$

The acceleration may be positive or negative.

Cor. II. When x and y are independent variables, $\Delta x = dx$, and $\Delta y = dy$.

The independent variable may be supposed to change in any manner whatever; its increments are arbitrary, and are themselves independent variables dependent not even on the value of the independent variable itself, while the increments of the dependent variable depend on both the independent variable and its increments. This arbitrary character of the independent variable leaves us free to make the most convenient supposition with reference to the manner of its variation, which is that this variation is uniform, or that its increments have no acceleration but are differentials.

Cor. III. The differential of a function at any value is what its succeeding increment would be if at that value its change became proportional to that of the increment of the independent variable. That is, when $y = f(x)$, (1) the limit of $\dfrac{\Delta y}{dy}$, as h approaches 0, is 1, and (2) $dy \propto h$. For example, in Fig. 4, we have (1) the limit of $\dfrac{DP'}{Dt}$, as h approaches 0, is 1, and (2) $Dt \propto PD$.

Since the differential of the function varies as the increment of the independent variable, the former will vary uniformly when the latter does.

Cor. IV. The differential of a function is positive or negative according as the function is increasing or decreasing.

Cor. V. In the increment $\Delta y = m_1 h + m_2 h^2$, $dy = m_1 h = m_1 dx$, and $ay = m_2 h^2$. Hence in Fig. 4, since $\Delta y = DP'$, and $dy = Dt$, we have $ay = tP'$.

26. In the equation $dy = m_1 dx$, m_1 or $\dfrac{dy}{dx}$ is called the Derivative or Differential Coefficient of y with respect to x, and is equal to $\tan XTP$, Fig. 4.

27. Since the limit of $\frac{\Delta y}{\Delta x}$, as Δx approaches 0, $= m_1$, Art. 24, and since $\frac{dy}{dx} = m_1$, we have

$$\underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta y}{\Delta x}\right] = \frac{dy}{dx}, \quad \text{or} \quad \frac{Dt}{PD}, \text{ Fig. 4.}$$

which is read, "the limit of $\frac{\Delta y}{\Delta x}$, as Δx approaches 0, is equal to $\frac{dy}{dx}$." We are to understand by this that the ratio of the proportional variations of y and x can be found by taking the limit of $\frac{\Delta y}{\Delta x}$. The student should note that the limits of Δy and Δx are not dy and dx; but the limit of $\frac{\Delta y}{\Delta x}$ is equal to $\frac{dy}{dx}$, because each is equal to m_1, just as $\frac{14}{21}$ is equal to $\frac{10}{15}$, because each is equal to $\frac{2}{3}$.

28. In finding the differential of a function it is not necessary to find the entire increment; only the part, or parts, involving the first power of h or dx, will be sufficient, for the terms which involve the higher powers of h form the acceleration of the function.

As an example let us find the differential of

$$u = x^4 - 5x^3 + 3x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Increasing x by h, we have

$$\begin{aligned} u + \Delta u &= (x + h)^4 - 5(x + h)^3 + 3(x + h) \\ &= x^4 + 4x^3h + \text{etc.} - 5(x^3 + 3x^2h + \text{etc.} + 3(x + h), \quad (2) \end{aligned}$$

where the omitted terms contain higher powers of h.

(2)−(1), $\qquad du = (4x^3 - 15x^2 + 3)dx,$ Ans.

29. Cor. I. If u and v are functions of x, when x is increased by h the proportional increments of u and v are du and dv, and these are the parts of Δu and Δv which involve the first power of h.

DIFFERENTIALS OF GEOMETRIC FUNCTIONS.

30. Differentiation is the operation of finding the differential of a function in terms of the differential of its variable. The process consists in finding the increment of the function and removing from it the acceleration, or in determining what the entire increment would be if it varied as the increment of the variable.

The following important formulas are deduced at this time more especially for the purpose of illustrating the preceding principles.

31. Differential of Plane Areas in Rectangular Co-ordinates. Let APP' be any plane curve, $OB = x$, $BP = y$, and area of $OBPA = u$; it is required to find the differential (du) of u.

When x is increased by $BC\ (= h)$, we have

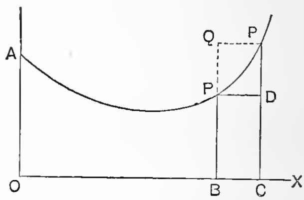

FIG. 5.

$$\Delta u = BCP'P = BCDP + PDP',$$

which corresponds in form with $\Delta y = m_1 h + m_2 h^2$, Art. 24, in which $m_1 h = BCDP$, $m_2 h^2 = PDP'$, $m_1 = BP = y$, and $m_2 h^2$ vanishes with h.

Since the initial side of Δu is $BP\ (= y)$, the rectangle $BCDP$ $(= yh)$ is what Δu would be had it varied as h; hence the area of the rectangle is the differential of u.

$$\therefore\ du = yh \quad \text{or} \quad ydx.$$

Or thus: since the limit of $BCP'P \div BCDP$, as BC approaches 0, is 1, and since $BCDP \propto BC$, $BCDP$ $(= ydx)$ is the differential of $OBPA$ $(= u)$.

Method by Limits. The increment $BCP'P$ is $>$ $BCDP$ and $<$ $BCP'Q$; that is,

$$y\Delta x < \Delta u < (y + \Delta y)\Delta x; \quad \therefore \; y < \frac{\Delta u}{\Delta x} < y + \Delta y.$$

As Δx approaches 0, the limit of Δy is 0, and that of $\dfrac{\Delta u}{\Delta x}$ *is equal to* $\dfrac{du}{dx}$; hence we have

$$y < \frac{du}{dx} < y; \quad \text{that is, } \frac{du}{dx} = y, \quad \text{or} \quad du = ydx.$$

Hence, *the required differential is equal to the value of the ordinate expressed in terms of x, multiplied by the differential of the abscissa.*

Cor. I. In Fig. 5, au, the acceleration of u, is the area of PDP'.

32. Differentials of Solids of Revolution in Rectangular Co-ordinates. In Fig. 5, let v equal the volume of the solid generated by the revolution of $OBPA$ about OX as an axis; it is required to find the differential of v.

When x is increased by h the corresponding increment of v is the volume of the solid generated by revolving $BCP'P$ about BC as an axis; that is,

$$\begin{aligned} \Delta v &= \text{vol. generated by } BCP'P \\ &= \text{vol. gen'd by } BCDP + \text{vol. gen'd by } PDP' \\ &= \pi y^2 h + \text{vol. gen'd by } PDP'. \end{aligned}$$

Since the initial base of Δv is the circle whose radius is y, the cylinder generated by $BCDP$ $(=\pi y^2 h)$ is what the increment Δv would be had it varied as h.

$$\therefore \qquad dv = \pi y^2 h \quad \text{or} \quad \pi y^2 dx.$$

Method by Limits.

Vol. gen'd by $BCDP < \Delta v <$ vol. gen'd by $BCP'Q$,

$$\text{or} \qquad \pi y^2 \Delta x < \Delta v < \pi (y + \Delta y)^2 \Delta x.$$

$$\therefore \qquad \pi y^2 < \frac{\Delta v}{\Delta x} < \pi (y + \Delta y)^2.$$

Passing to limits, as in previous examples, we obtain

$$dv = \pi y^2 dx.$$

Hence, *the required differential is π times the square of the value of the ordinate expressed in terms of x, multiplied by the differential of the abscissa.*

Cor. I. In Fig. 5, av, the acceleration, is the volume generated by PDP'.

33. Differential of the Arc of a Plane Curve in Rectangular Co-ordinates. In Fig. 4, let $s =$ the length of the arc AP, then $PP' = \Delta s$. Since Dt is what Δy would be had it varied as h, Pt is what Δs would be had it varied as h; hence, $Dt = dy$ and $Pt = ds$; and since $Pt = \sqrt{PD^2 + Dt^2}$, we have

$$ds = \sqrt{dx^2 + dy^2}, \quad \text{or} \quad \left(\sqrt{1 + \frac{dy^2}{dx^2}}\right) dx.$$

Hence, *the required differential is the square root of the sum of one and the square of the value of the derivative of y with respect to x, expressed in terms of x, multiplied by the differential of x.*

COR. I. *The limit of the ratio of an arc of any plane curve to its chord is unity.*

For (Fig. 4),

$$\frac{\text{arc } PP'}{\text{chord } PP'} = \frac{\Delta s}{\sqrt{\overline{\Delta x}^2 + \overline{\Delta y}^2}} = \frac{\dfrac{\Delta s}{\Delta x}}{\sqrt{1 + \left(\dfrac{\Delta y}{\Delta x}\right)^2}},$$

the limit of which is evidently unity.

34. Differential of Surfaces of Revolution in Rectangular Co-ordinates. Let $s =$ the length of the arc AP, and $S =$ the area of the surface generated by the revolution of AP about OX as an axis, then $\Delta S =$ the area of the surface generated by the revolution of PP' $(= \Delta s)$ about BC $(= h)$.

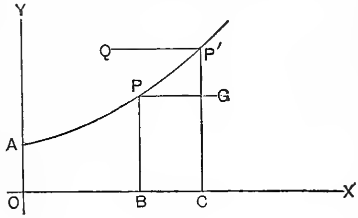

FIG. 6.

At P and P' draw PG and $P'Q$ each equal to PP' and parallel to OX; the areas of the surfaces generated by revolving PG and $P'Q$ about OX are $2\pi y\Delta s$ and $2\pi(y + \Delta y)\Delta s$, and evidently ΔS lies between the former and the latter. That is,

$$2\pi y\Delta s < \Delta S < 2\pi(y + \Delta y)\Delta s.$$

$$\therefore \qquad 2\pi y < \frac{\Delta S}{\Delta s} < 2\pi(y + \Delta y).$$

As h approaches 0, the limit of Δy is 0, and the limit of $\dfrac{\Delta S}{\Delta s}$ is equal to $\dfrac{dS}{ds}$; hence we have

$$2\pi y < \frac{dS}{ds} < 2\pi y, \quad \text{or} \quad dS = 2\pi y ds,$$

$$\text{or} \quad dS = 2\pi y \sqrt{dx^2 + dy^2}.$$

Hence, *the required differential is the product of the circumference of a circle whose radius is y, by the differential of the arc of the generating curve.*

35. Differential of Plane Areas in Polar Co-ordinates. Let APP' be any plane curve, and let O be the pole, OP $(= r)$ the radius vector, and put $\theta = XOP$ and $u =$ the area of OAP; it is required to find the differential of u.

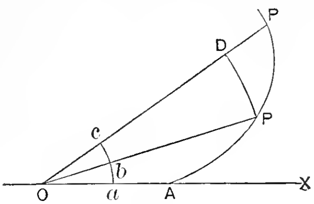

FIG. 7.

When θ is increased by the angle POP' $(= \Delta\theta)$, u will be increased by the area of POP' $(= \Delta u)$.

From O as a centre with the radius Oa $(= 1)$ describe the arc bc $(= \Delta\theta$ or $d\theta)$, and with the radius OP $(= r)$ describe the arc PD $(= rd\theta)$.

Since OP is the initial side of Δu, the area of the sector POD is what Δu would be had it varied as $\Delta\theta$.

$$\therefore \qquad du = \tfrac{1}{2}OP \times PD = \tfrac{1}{2}r^2 d\theta.$$

Or thus: since the limit of $\dfrac{POP'}{POD}$, as bc approaches 0, is 1, and since $POD \propto bc$, POD $(= \frac{1}{2}r^2 d\theta)$ is the differential of OAP $(= u)$.

COR. I. The acceleration of u is the area of PDP'.

CHAPTER II.

ELEMENTARY DIFFERENTIATION AND INTEGRATION.

DIFFERENTIATION.

36. Every function may be differentiated by the principles heretofore established, but in practice it is better to use rules, which we now proceed to deduce.

37. To differentiate a constant. Since a constant has no increment, its differential is 0. That is,

$$dc = 0, \quad \text{and} \quad dC = 0.$$

Hence, *the differential of a constant is* 0.

38. To differentiate the product of a constant by a variable.

Let $$u = cv, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where c is a constant and v is a function of x.

Let h represent any variable increment of x estimated from any particular value of x, and du and dv the corresponding proportional increments of u and v, Art. 29; then

$$u + du = c(v + dv) = cv + cdv. \quad . \quad . \quad . \quad . \quad (2)$$

$(2) - (1),$ $$du = cdv. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$\therefore$ RULE.—*Multiply the constant by the differential of the variable.*

39. To differentiate the product of two variables.

Let $$u = vy, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where v and y are functions of x.

Let x' represent any particular value of x, and u', v', y' the corresponding values of u, v, and y; then

$$u' = v'y'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let h represent any variable increment of x estimated from x', and du, dv, and dy the corresponding proportional increments of u, v, and y (Art. 29); then

$$\begin{aligned} u' + du &= (v' + dv)(y' + dy) - (dv)(dy) \\ &= v'y' + y'dv + v'dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2) \end{aligned}$$

The term $(dv)(dy)$ is eliminated, or dropped from the second side, since it varies disproportionally with h (Art. 17) and is therefore a part of the acceleration of u. Subtracting (1) from (2) we have

$$du = y'dv + v'dy.$$

Since y' and v' are any corresponding values of y and v, we have

$$du = d(vy) = vdy + ydv.$$

∴ RULE.—*Multiply the first by the differential of the second, and the second by the differential of the first, and add the two products.*

40. To differentiate the product of any number of variables. Let $u = vyz$, where v, y, and z are functions of x.

Assume $\qquad w = yz,$

therefore $\qquad u = vw;$

then $\qquad du = (w)dv + v(dw),$

also $\qquad dw = zdy + ydz;$

therefore $\qquad du = (yz)dv + v(zdy + ydz)$

$$= yzdv + vzdy + vydz.$$

In a similar manner it may be shown that

$$d(vwyz) = vwydz + vwzdy + vyzdw + wyzdv.$$

∴ RULE.—*Take the sum of the products obtained by multiplying the differential of each by all the other variables.*

41. To differentiate a fraction.

Let $u = \frac{v}{y}$, where v and y are functions of x.

Since $u = \frac{v}{y}$, we have $uy = v$.

Differentiating, $\quad udy + ydu = dv.$ (Art. 39)

Whence,
$$du = \frac{dv - udy}{y} = \frac{dv - \frac{v}{y}dy}{y}$$
$$= \frac{ydv - vdy}{y^2}.$$

∴ RULE.—*Multiply the denominator by the differential of the numerator and from the product subtract the numerator multiplied by the differential of the denominator, and divide the result by the square of the denominator.*

42. COR. I. The differential of a fraction whose numerator is a constant is minus the numerator into the differential of the denominator divided by the square of the denominator.

For, Art. 41, $d\left(\frac{c}{y}\right) = \frac{ydc - cdy}{y^2} = -\frac{cdy}{y^2}$, since $dc = 0$.

43. To differentiate a variable having a constant exponent.

Let $u = v^n$, where v is any function of x, and n is any constant integer or fraction.

I. When the exponent is a positive integer.

Since $\quad v^n = v \cdot v \cdot v \ldots$ to n factors,

$$d(v^n) = v^{n-1}dv + v^{n-1}dv \ldots \text{ to } n \text{ terms} \qquad \text{(Art. 40)}$$
$$= nv^{n-1}dv.$$

II. When the exponent is a positive fraction.

Let $u = v^{\frac{a}{c}}$, then $\qquad u^c = v^a.$

Differentiating, $\qquad cu^{c-1}du = av^{a-1}dv.$

Substituting for u, $\qquad cv^{\left(\frac{a}{c}\right)c-1}du = av^{a-1}dv.$

Reducing, $\qquad cv^{a-\frac{a}{c}}du = av^{a-1}dv.$

Dividing by $cv^{a-\frac{a}{c}}$, $\qquad du = \frac{a}{c}v^{\frac{a}{c}-1}dv.$

III. When the exponent is negative.

Let $u = v^{-m}$, then $\qquad u = \frac{1}{v^m}.$

Differentiating, Art. 42, $du = -\frac{mv^{m-1}}{v^{2m}}dv$

$$= -mv^{-m-1}dv.$$

Hence, for all the cases, we have this

RULE.—*Take the product of the exponent, the variable with its exponent diminished by 1, and the differential of the variable.*

Thus: $\qquad d(x^5) = 5x^4dx, \qquad d(u^{-7}) = -7u^{-8}du,$

$$d(v^{\frac{5}{4}}) = \tfrac{5}{4}v^{\frac{1}{4}}dv, \qquad d(z^{-\frac{2}{3}}) = -\tfrac{2}{3}z^{-\frac{5}{3}}dz,$$

$$d(y^{\frac{1}{3}}) = \tfrac{1}{3}y^{-\frac{2}{3}}dy = \frac{dy}{3\sqrt[3]{y^2}}.$$

The same rule holds when the exponent n is irrational or imaginary.

44. COR. I. The differential of the square root of a variable is the differential of the variable divided by twice the radical.

For, $\quad d(\sqrt{v}) = d(v^{\frac{1}{2}}) = \frac{1}{2}v^{-\frac{1}{2}}dv = \dfrac{dv}{2\sqrt{v}}.$

45. To differentiate the algebraic sum of several variables.

Let $$u = v + y - z, \quad \ldots \ldots \quad (1)$$

where v, y, and z are functions of x.

Let h represent any variable increment of x estimated from any particular value of x, and du, dv, dy, and dz the corresponding proportional increments of u, v, y, and z; then

$$u + du = v + dv + y + dy - (z + dz). \quad . \; . \quad (2)$$

$(2) - (1)$, $\quad du = dv + dy - dz.$

$\therefore$ RULE.—*Take the algebraic sum of their differentials.*

EXAMPLES.

Differentiate the following:

1. $y = x^3 + 5x^2 - 3x + 7.$

$$\begin{aligned} dy &= d(x^3) + d(5x^2) - d(3x) + d(7) && \text{(Art. 45)} \\ &= d(x^3) + 5d(x^2) - 3d(x) + d(7) && \text{(Art. 38)} \\ &= 3x^2dx + 10xdx - 3dx + 0 && \text{(Arts. 37, 43)} \\ &= (3x^2 + 10x - 3)dx, \text{ Ans.} \end{aligned}$$

2. $y = x^2 + 5x + 3.$ $\quad dy = (2x + 5)dx.$
3. $y = x^4 - 4x^3 - 3x^2.$ $\quad dy = (4x^3 - 12x^2 - 6x)dx.$
4. $y = 3x^5 - 7x^2 - 9.$ $\quad dy = (15x^4 - \text{etc.})dx.$
5. $y = x^6 + 2x^5 - 7x + 10.$ $\quad dy = (6x^5 + 10x^4 + \text{etc.})dx.$
6. $y = x^2 + x^{-2}.$ $\quad dy = (2x - 2x^{-3})dx.$
7. $y = x^{-3} + x^{\frac{1}{2}}.$ $\quad dy = (-3x^{-4} + \frac{1}{2}x^{-\frac{1}{2}})dx.$
8. $y = x^{\frac{5}{3}} - ax^{-4} + b.$ $\quad dy = (\frac{5}{3}x^{\frac{2}{3}} + 4ax^{-5})dx.$
9. $y = 6x^{\frac{7}{2}} - 4x^{\frac{5}{2}} + 2x^{\frac{3}{2}}.$ $\quad dy = (21x^{\frac{5}{2}} - \text{etc.})dx.$
10. $y = 3x^{-\frac{1}{3}} - 6x^{-\frac{2}{3}} + 5.$ $\quad dy = (-x^{-\frac{4}{3}} + 4x^{-\frac{5}{3}})dx.$
11. $y = ax^m - bx^{-n}.$ $\quad dy = (max^{m-1} + bnx^{-n-1})dx.$
12. $y = \sqrt{3x} + 4\sqrt{z}.$ $\quad dy = \dfrac{3dx}{2\sqrt{3x}} + \dfrac{2dz}{\sqrt{z}}.$

13. $y = \sqrt[3]{x} + \dfrac{1}{x}.$ $(= x^{\frac{1}{3}} + x^{-1}.)$ $dy = (\frac{1}{3}x^{-\frac{2}{3}} - x^{-2})dx.$

14. $y = \dfrac{a}{x} + \dfrac{b}{x^2}.$ $(= ax^{-1} + bx^{-2}.)$ $dy = \left(-\dfrac{a}{x^2} - \dfrac{2b}{x^3}\right)dx.$

15. $y^2 = 4ax.$ $dy = \dfrac{2a}{y}\,dx.$

16. $y^2 + x^2 = R^2.$ $dy = -\dfrac{x}{y}\,dx.$

17. $a^2y^2 + b^2x^2 = a^2b^2.$ $dy = -\dfrac{b^2x}{a^2y}dx.$

Find the following:

18. $d(x^2 + 1)(x^2 + 2x).$

$$(x^2 + 1)d(x^2 + 2x) + (x^2 + 2x)d(x^2 + 1) \qquad \text{(Art. 39)}$$
$$= (x^2 + 1)(2x + 2)dx + (x^2 + 2x)2xdx$$
$$= (2x^3 + 2x^2 + 2x + 2)dx + (2x^3 + 4x^2)dx$$
$$= (4x^3 + 6x^2 + 2x + 2)dx, \text{ Ans.}$$

19. $d(x - 1)(x^2 + x + 1).$ $3x^2dx.$
20. $d(ax^4y^3).$ $4ax^3y^3dx + 3ax^4y^2dy.$
21. $d(x)(y^2 + 1).$ $(y^2 + 1)dx + 2xydy.$
22. $d(x^2 - 1)(x^4 + 1).$ $(6x^5 - 4x^3 + 2x)dx.$
23. $d(6x^{\frac{1}{2}}y^{\frac{1}{3}}).$ $3x^{-\frac{1}{2}}y^{\frac{1}{3}}dx + 2x^{\frac{1}{2}}y^{-\frac{2}{3}}dy.$
24. $d[x^{-2}(1 + x^{-3})].$ $(-2x^{-3} - 5x^{-6})dx.$
25. $d(1 + x)(x + x^2)x.$
26. $d(1 + 2x^2)(1 + 4x^3).$
27. $d(x + 1)(x^3 - x^2 + x - 1).$ $4x^3dx.$
28. $d(x^3 + a)(3x^2 + b).$ $(15x^4 + 3bx^2 + 6ax)dx.$
29. $d(12x^{\frac{3}{2}}y^{\frac{4}{3}} + 15abc).$ $18x^{\frac{1}{2}}y^{\frac{4}{3}}dx + 16x^{\frac{3}{2}}y^{\frac{1}{3}}dy.$

Differentiate the following:

30. $u = \dfrac{1 + x}{1 - x}.$

$$du = \frac{(1-x)d(1+x) - (1+x)d(1-x)}{(1-x)^2} \qquad \text{(Art. 41)}$$

$$= \frac{2dx}{(1-x)^2}, \text{ Ans.}$$

31. $u = \dfrac{a-x}{x}$. $\qquad du = -\dfrac{adx}{x^2}$.

32. $u = \dfrac{1+x}{1+x^2}$. $\qquad du = \dfrac{(1-2x-x^2)dx}{(1+x^2)^2}$.

33. $u = \dfrac{x^3}{(1+x)^3}$. $\qquad du = \dfrac{3x^2dx}{(1+x)^4}$.

34. $u = \dfrac{x^3}{x^2-1} - \dfrac{x^2}{x-1}$. $\qquad du = \dfrac{2xdx}{(x^2-1)^2}$.

35. $u = \dfrac{x^3}{a^2-x^2}$.

36. $u = \dfrac{2x^2-3}{4x+x^2}$. $\qquad du = \dfrac{8x^2+6x+12}{(4x+x^2)^2}dx$.

37. $u = \dfrac{3}{(a+bx)^3}$.

38. $u = \dfrac{x^3}{(1+x)^2}$. $\qquad du = \dfrac{3x^2+x^3}{(1+x)^3}dx$.

39. $u = \sqrt{x^2-3x+5}$.

$$du = \frac{d(x^2-3x+5)}{2\sqrt{x^2-3x+5}} = \frac{(2x-3)dx}{2\sqrt{x^2-3x+5}}. \qquad \text{(Art. 44)}$$

40. $u = \sqrt{3x^2-5}$. $\qquad du = \dfrac{3xdx}{\sqrt{3x^2-5}}$.

41. $u = \sqrt{4x} + \sqrt{9x^3}$. $\qquad du = \dfrac{2+9x}{2\sqrt{x}}dx$.

42. $u = x\sqrt{1+x}$. $\qquad du = \dfrac{2+3x}{2\sqrt{1+x}}dx$.

43. $u = \dfrac{x}{\sqrt{1-x^2}}$. $\qquad du = \dfrac{dx}{(1-x^2)^{\frac{3}{2}}}$.

44. $u = \sqrt{\dfrac{1+x}{1-x}}$. $\qquad du = \dfrac{dx}{(1-x)\sqrt{1-x^2}}$.

45. $u = (x^3 - 3x^2 + 4x - 5)^5.$

$$du = 5(x^3 - 3x^2 + 4x - 5)^4 d(x^3 - 3x^2 + 4x - 5) \qquad \text{(Art. 43)}$$
$$= 5(x^3 - 3x^2 + 4x - 5)^4 (3x^2 - 6x + 4)dx.$$

46. $u = (x^3 - 7x + 5)^3.$ $\qquad du = 3(x^3 - 7x + 5)^2(3x^2 - 7)dx.$

47. $u = (x^2 + 5x - 9)^{\frac{4}{3}}.$ $\qquad du = \frac{4}{3}(x^2 + 5x - 9)^{\frac{1}{3}}(2x + 5)dx.$

48. $u = (2\sqrt{x} + 3)^{-3}.$ $\qquad du = -\dfrac{3dx}{\sqrt{x}(2\sqrt{x} + 3)^4}.$

49. $u = \left(\dfrac{x}{1 - x}\right)^m.$ $\qquad du = \dfrac{mx^{m-1}dx}{(1 - x)^{m+1}}.$

50. $u = (a + x)\sqrt{a - x}.$ $\qquad du = \dfrac{(a - 3x)dx}{2\sqrt{a - x}}.$

51. $y = \dfrac{3x^3 + 2}{x(x^3 + 1)^{\frac{2}{3}}}.$ $\qquad \dfrac{dy}{dx} = -\dfrac{2}{x^2(x^3 + 1)^{\frac{5}{3}}}.$

52. $y = \dfrac{\sqrt{(x + a)^3}}{\sqrt{x - a}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{(x - 2a)\sqrt{x + a}}{(x - a)^{\frac{3}{2}}}.$

SLOPE OF CURVES.

46. Direction. The direction of a straight line is determined by its angle of direction, which is the angle formed by the axis of x and the line.

The **Slope** of a line is the tangent of its angle of direction.

Thus, in Fig. 4, the angle of direction of the line TP is XTP, and the slope of the line is $\tan XTP = \dfrac{Dt}{PD} = \dfrac{dy}{dx}.$

47. The direction of a curve at any point is the same as that of a tangent to the curve at that point.

For, at the point P (Fig. 4), the deviation of the secant SPP' from the curve PP', arising from the former's cutting the latter, diminishes indefinitely as P' approaches P; and since the tangent TPt, which touches the curve at P, is the limiting position of the secant, the tangent has the same direction as the curve at the point P or (x, y).

48. Cor. I. The slope of a curve at any point is the slope of its tangent at that point. Therefore the slope of a curve at the point (x, y) is $\frac{dy}{dx}$. The differential of the arc of a curve at any point is a straight line laid off on the tangent to the curve at that point, Art. 33.

49. The angle of direction of a curve at the point (x, y) is usually denoted by ϕ; that is, $\phi = XTP$, Fig. 4.

$$\therefore \tan\phi = \frac{dy}{dx}, \quad \sin\phi = \frac{dy}{ds}, \quad \text{and} \quad \cos\phi = \frac{dx}{ds},$$

where $ds =$ the differential of the arc of the curve, Art. 33.

EXAMPLES.

1. The equation of a curve is $y = x^3 - 2x$; find the slope of the curve at the point (x,y).

Differentiating the equation and dividing by dx, we have

$$\frac{dy}{dx} = 3x^2 - 2, \text{ Ans.}$$

2. In the same curve find the slope at the point where $x = 1$.

$$\frac{dy}{dx} = 1.$$

3. Find the slope of the parabola $y^2 = 9x$ at the point where $x = 4$.

$$\frac{dy}{dx} = \frac{3}{4}.$$

4. In the same parabola find the point where the curve makes an angle of 45° with the axis of x.

Since $\tan 45° = 1$, we make $\frac{dy}{dx} = \frac{9}{2y} = 1$; hence $2y = 9$, or $y = 4\frac{1}{2}$, and $x = \frac{y^2}{9} = \frac{9}{4} = 2\frac{1}{4}$.

5. In the circle $y^2 + x^2 = R^2$, find the point where the slope of the curve is $-\frac{3}{4}$. Where $y = \frac{4}{5}R$, and $x = \frac{3}{5}R$.

6. In the same circle find the point where the curve is parallel with the line whose equation is $5y + 12x = 60$.

Where $x = \frac{12}{13}R$.

7. At what angles does the parabola $y^2 = 6x$ cut the circle $y^2 + x^2 = 16$?

Find their slopes at their points of intersection; then find the angles between the lines having these slopes. Thus: solving the two equations, we find (for one of the points of intersection of the two curves) $x = 2$ and $y = 2\sqrt{3}$. The slope of the parabola at this point is $\dfrac{3}{y}$ $(= \frac{1}{2}\sqrt{3})$, and that of the circle is $-\dfrac{x}{y}$ $(= -\frac{1}{3}\sqrt{3})$. Therefore the tangent of the required angle is

$$\frac{\frac{1}{2}\sqrt{3} + \frac{1}{3}\sqrt{3}}{1 - \frac{1}{2}\sqrt{3} \times \frac{1}{3}\sqrt{3}} = \frac{\frac{5}{6}\sqrt{3}}{1 - \frac{1}{2}} = \frac{5}{3}\sqrt{3} = 2.88 +.$$

8. At what angles does the line $3y - 2x - 8 = 0$ cut the parabola $y^2 = 8x$?

$\text{Tan}^{-1}\, .2$ and $\tan^{-1} .125$.

9. The equation of a curve is $y = x^3 - 9x^2 + 24x - 11$. (1) Find the slope of the curve at the point where $x = 3$. (2) Find the values of x at the points where the slope of the curve is 45. (3) Find the values of x at the points where the curve is parallel with the axis of x.

(1) -3; (2) $x = 7$ and -1; (3) $x = 2$ and 4.

10. Find the point where the curve $y = x^2 - 7x + 3$ is parallel to the line $y = 5x + 2$.

Where $x = 6$.

11. Find the point where the parabola $y^2 = 4ax$ is parallel with the circle $y^2 = 2Rx - x^2$.

Where $x = R - 2a$.

12. Show that the ellipse $\dfrac{x^2}{18} + \dfrac{y^2}{8} = 1$ intersects the hyperbola $x^2 = y^2 + 5$ at right angles.

13. At what angles does the circle $x^2 + y^2 = 8ax$ intersect the cissoid $y^2 = \dfrac{x^3}{2a - x}$?

At the origin, 90°; at the other two points, 45°.

INTEGRATION.

50. Integration is the inverse of differentiation. Thus, while differentiation is the process of deriving the differential of a function from the function, integration is the process of deriving the function from its differential. The function is called the integral of the differential. Thus, the differential of x^3 being $3x^2dx$, x^3 is the integral of $3x^2dx$.

The **Sign** of integration is $\int$; thus $\int 3x^2dx$ indicates the operation of integrating $3x^2dx$, therefore $\int 3x^2dx = x^3$, which is read "the integral of $3x^2dx$ is x^3." The two signs d and $\int$ annul each other;* $\int d(x^3) = x^3$ and $d\int(3x^2dx) = 3x^2dx$.

51. Dependent Integration. When the process of integrating depends on reversing the corresponding process of differentiating, as is usually the case, it is called dependent integration. This process will now be employed in establishing rules for integrating elementary differentials.

52. To integrate 0.

Since $dC = 0$, Art. 37, we have $\int 0 = C$.

Therefore, as 0 may be added to any differential without changing its value, the general form of its integral will not be complete without an indeterminate constant term. This constant term, as will be seen, may be eliminated, or determined from the data of any particular problem.

53. To integrate the product of a differential and a constant.

Since $d(cv) = cdv$, Art. 38, we have $\int cdv = c\int dv = cv + C$.

---

* To be exact, $d\int f(x)dx = f(x)dx$, but $\int df(x) = f(x) + C$.

$\therefore$ RULE.—*Integrate the differential, and multiply the result by the constant.*

54. To integrate $v^n dv$, where n has any positive or negative, integral or fractional value except -1.

Since $d\left(\dfrac{v^{n+1}}{n+1}\right) = v^n dv$, Art. 43, we have

$$\int v^n dv = \frac{v^{n+1}}{n+1} + C.$$

$\therefore$ RULE.—*Remove dv, increase the exponent by 1, and divide the result by the new exponent.*

Thus, $\int x^4 dx = \frac{1}{5}x^5 + C;$ $\qquad \int y^{\frac{2}{3}} dy = \frac{3}{5}y^{\frac{5}{3}} + C;$

$$\int v^{-4} dv = -\tfrac{1}{3}v^{-3} + C; \quad \int 3u^{-\frac{1}{2}} du = 6u^{\frac{1}{2}} + C.$$

55. Cor. I. The above rule applies also to $\int (v)^n dv$, where v is any function of a variable.

Thus, $\int (x^2 - 3x + 5)^3 [2x - 3] dx = \frac{1}{4}(x^2 - 3x + 5)^4 + C.$

In this example $v = x^2 - 3x + 5$, and $dv = (2x - 3)dx$; that is, the rule applies whenever the factor without the ( ), viz., $[2x - 3]dx$, is the differential of the quantity within the ( ).

56. Cor. II. $\int v^n dv = \frac{1}{c}\int (v)^n c\,dv$; introducing a constant thus is sometimes necessary to render the quantity without the ( ) the differential of the one within. Thus,

$$\int (4x + 2x^2)^{\frac{2}{3}}(1 + x)dx = \tfrac{1}{4}\int (4x + 2x^2)^{\frac{2}{3}}(4 + 4x)dx$$
$$= \tfrac{3}{20}(4x + 2x^2)^{\frac{5}{3}} + C.$$

57. Cor. III. Any differential of the form $(a + bx^n)^p x^m dx$, where $n = m + 1$, can be integrated by the above rule; thus,

$$\int (a + bx^n)^p x^{n-1} dx = \frac{1}{nb}\int (a + bx^n)^p nbx^{n-1} dx$$

$$= \frac{1}{nb}\int (a + bx^n)^p d(a + bx^n) = \frac{(a + bx^n)^{p+1}}{nb(p + 1)} + C.$$

Thus, $\int x\sqrt{3 + 5x^2}\, dx = \int (3 + 5x^2)^{\frac{1}{2}} x dx = \frac{1}{15}(3 + 5x^2)^{\frac{3}{2}} + C.$

$$\int \frac{x^2 dx}{(5 + 7x^3)^4} = \int (5 + 7x^3)^{-4} x^2 dx = -\frac{1}{63}(5 + 7x^3)^{-3} + C.$$

The rule, Art. 54, does not hold for $n = -1$, for the reason that $d(v^0)$ is not $v^{-1}dv$, but 0. The formula for this case will be derived subsequently, Art. 180, formula 2.

58. To integrate the algebraic sum of two or more differentials.

Since $$d(u + v - z) = du + dv - dz, \qquad \text{(Art. 45)}$$

we have $$\int (du + dv - dz) = \int du + \int dv - \int dz$$
$$= u + v - z + C.$$

∴ Rule.—*Take the algebraic sum of their integrals.*

Thus, $$\int (3x^2 - 2x + 5)dx = \int 3x^2 dx - \int 2x dx + \int 5dx$$
$$= x^3 - x^2 + 5x + C.$$

EXAMPLES.

Integrate the following:

1. $dy = (6x^2 - 4x + 5)dx.$

$$y = \int 6x^2 dx - \int 4x dx + \int 5dx \qquad \text{(Art. 58)}$$
$$= 6\int x^2 dx - 4\int x dx + 5\int dx \qquad \text{(Art. 53)}$$
$$= 6 \cdot \frac{x^3}{3} - 4 \cdot \frac{x^2}{2} + 5 \cdot x + C \qquad \text{(Art. 54)}$$
$$= 2x^3 - 2x^2 + 5x + C, \text{ Ans.}$$

2. $dy = (7x^6 - 5x^4)dx.$ $\qquad y = x^7 - x^5 + C.$

3. $dy = (5x^2 - 3x^{-4})dx.$ $\qquad y = \frac{5}{3}x^3 + x^{-3} + C.$

4. $dy = (3x^{\frac{1}{2}} + 2x^{-3})dx.$ $\qquad y = 2x^{\frac{3}{2}} - x^{-2} + C.$

5. $dy = (x^{\frac{1}{3}} + x^{\frac{1}{2}})dx.$ $\qquad y = \frac{3}{4}x^{\frac{4}{3}} + \frac{2}{3}x^{\frac{3}{2}} + C.$

6. $dy = \dfrac{dx}{\sqrt[3]{x}}$ or $x^{-\frac{1}{3}}dx.$ $\qquad y = \frac{3}{2}x^{\frac{2}{3}} + C.$

7. $dy = \dfrac{dx}{x^4}$ or $x^{-4}dx.$ $\qquad y = -\frac{1}{3}x^{-3} + C.$

8. $dy = \left(ax^2 + \dfrac{1}{2\sqrt{x}}\right)dx.$ $\qquad y = \frac{1}{3}ax^3 + \sqrt{x} + C.$

9. $dy = \left(\dfrac{a}{x^4} - \dfrac{b}{x^3} + \dfrac{c}{x^2}\right)dx.$ $\qquad y = -\dfrac{a}{3x^3} +$ etc.

10. $dy = (1 + x^4)^3x^3dx.$

$$y = \int \tfrac{1}{4}(1 + x^4)^3 4x^3dx = \tfrac{1}{4}\int (1 + x^4)^3 d(1 + x^4) \quad \text{(Art. 56)}$$

$$= \tfrac{1}{16}(1 + x^4)^4 + C, \text{ Ans.} \quad \text{(Art. 55)}$$

11. $dy = (1 + x)^3dx.$ $\qquad y = \frac{1}{4}(1 + x)^4 + C.$

12. $dy = (1 + x)^{\frac{2}{3}}dx.$ $\qquad y = \frac{3}{5}(1 + x)^{\frac{5}{3}} + C.$

13. $dy = (1 - x)^{-2}dx.$ $\qquad y = (1 - x)^{-1} + C.$

14. $dy = (a + x^2)^{\frac{1}{2}}xdx.$ $\qquad y = \frac{1}{3}(a + x^2)^{\frac{3}{2}} + C.$

15. $dy = (a - x^3)^{-\frac{1}{2}}x^2dx.$ $\qquad y = -\frac{2}{3}(a - x^3)^{\frac{1}{2}} + C.$

16. $dy = (b + x^{-2})^{\frac{1}{2}}x^{-3}dx.$ $\qquad y = -\frac{1}{3}(b + x^{-2})^{\frac{3}{2}} + C.$

Find the following:

17. $\int (x^4 - 3x^{-4})dx.$ $\qquad \frac{1}{5}x^5 + x^{-3} + C.$

18. $\int \dfrac{xdx}{\sqrt{1 - x^2}} = \int (1 - x^2)^{-\frac{1}{2}}xdx.$ $\qquad -\sqrt{1 - x^2} + C.$

19. $\int \left(x + \dfrac{x^2}{\sqrt[3]{1 + x^3}}\right)dx.$ $\qquad \dfrac{x^2}{2} + \dfrac{1}{2}(1 + x^3)^{\frac{2}{3}} + C.$

20. $\int \left(\dfrac{\sqrt{a} + 3cx}{2\sqrt{x}}\right)dx.$ $\qquad \sqrt{ax} + c\sqrt{x^3} + C.$

21. $\int \left(\dfrac{2x - 3}{\sqrt{2x^2 - 6x + 5}}\right)dx.$ $\qquad \sqrt{2x^2 - 6x + 5} + C.$

22. $\int (x^3 + 3x^2 - 6)^{\frac{1}{2}}(x^2 + 2x)dx.$ $\qquad \frac{2}{9}(x^3 + 3x^2 - 6)^{\frac{3}{2}} + C.$

23. $\int \frac{xdx}{(1 - x^2)^{\frac{3}{2}}}.$ $\qquad \frac{1}{\sqrt{1 - x^2}} + C.$

24. $\int \frac{4aydy}{(b - 2y^2)^2}.$ $\qquad \frac{a}{b - 2y^2} + C.$

25. $\int \frac{6x^2dx}{\sqrt{4x^3 - 5}}.$ $\qquad \sqrt{4x^3 - 5} + C.$

26. $\int \frac{(x - 1)dx}{\sqrt{3x^2 - 6x + 7}}.$ $\qquad \frac{1}{3}\sqrt{3x^2 - 6x + 7} + C.$

27. $\int (1 - x^2)^3 dx.$ $\qquad x - x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7 + C.$

28. $\int \left(\frac{a + x^2}{x}\right)^2 dx.$ $\qquad -\frac{a^2}{x} + 2ax + \frac{x^3}{3} + C.$

29. $\int \left(\frac{a + bx + cx^2}{\sqrt{x}}\right)dx.$ $\qquad 2\sqrt{x}(a + \frac{1}{3}bx + \frac{1}{5}cx^2) + C.$

30. $\int \sqrt{x}(a^2 - x^2)^3 dx.$ $\qquad 2x^{\frac{3}{2}}\left(\frac{a^6}{3} - \frac{3a^4x^2}{7} + \frac{3a^2x^4}{11} - \frac{x^6}{15}\right) + C.$

The following differentials may be reduced to the form of $(a + bx^n)^p x^{n-1}dx$, and then integrated by Art. 57.

31. $\int \frac{x^{-2}dx}{\sqrt{a^2 + x^2}} = \int \frac{x^{-3}dx}{\sqrt{a^2x^{-2} + 1}}.$ $\qquad -\frac{\sqrt{a^2 + x^2}}{a^2x} + C.$

Multiply the binomial under the radical sign by x^{-2}, and the numerator of the fraction by x^{-1}.

32. $\int \frac{\sqrt{x^2 - a^2}\,dx}{x^4}.$ $\qquad \frac{(x^2 - a^2)^{\frac{3}{2}}}{3a^2x^3} + C.$

33. $\int \frac{\sqrt{2ax - x^2}\,dx}{x^3}.$ $\qquad -\frac{(2ax - x^2)^{\frac{3}{2}}}{3ax^3} + C.$

34. $\int \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}}.$ $\qquad \frac{x}{a^2\sqrt{a^2 + x^2}} + C.$

35. $\int \frac{xdx}{(2ax - x^2)^{\frac{3}{2}}}.$ $\qquad \frac{x}{a\sqrt{2ax - x^2}} + C.$

PROBLEMS.

59. The Problem of Integration is the inverse of that of differentiation; if the latter is "given a curve to find its slope," the former is "given the slope of a curve to find the curve"; if the latter is "given a function to find its rate of change," the former is "given the rate of change of a function to find the function." Since the general problem of differentiation is "given a function and its variable to find their proportional changes," that of integration is "given the proportional changes of a function and its variable to find the function."

60. Definite Value of the Constant C. To complete each integral as determined by the preceding rules, we have added a constant quantity C. While the value of this constant is unknown, it is said to be *indefinite;* but it becomes *definite* when its value is assigned or determined by the conditions of the problem under consideration. The signification of C and the manner of determining its definite value are illustrated in the following examples.

EXAMPLES.

1. Required the equation of the curve whose slope at the the point (x, y) is $4x^3 - 2x + 3$.

By Art. 48, the slope of a curve at (x, y) is $\frac{dy}{dx}$.

$$\therefore \qquad \frac{dy}{dx} = 4x^3 - 2x + 3,$$

or

$$dy = (4x^3 - 2x + 3)dx.$$

Integrating, $$y = x^4 - x^2 + 3x + C,$$

which is the indefinite integral or equation required.

To determine the value of C we may make $x = 0$, which gives $y_0 = C$, where y_0 indicates what y becomes when $x = 0$, and is therefore the distance from the origin to where the curve cuts the axis of y.

Therefore the equation may be written

$$y = x^4 - x^2 + 3x + y_0,$$

which becomes definite when y_0 is known.

When we know any corresponding values of an integral and its variable, C can be determined. Thus, if it is given that the last curve passes through the point $x' = 2$, $y' = 10$, then when $x = 2$, $y = 10$, and we have

$$10 = 2^4 - 2^2 + 3\ 2 + C, \quad \text{or} \quad C = -8.$$

2. What is the equation of a curve which passes through the origin, and whose slope at the point (x, y) is $(a + x)^2$?

Here $\frac{dy}{dx} = (a + x)^2$, or $dy = (a + x)^2 dx$.

Integrating, $y = \frac{1}{3}(a + x)^3 + C$.

Since the curve passes through the origin, $y_0 = 0$; hence, making $x = 0$, we have

$$0 = \tfrac{1}{3}a^3 + C; \quad \therefore\ C = -\tfrac{1}{3}a^3,$$

and the required equation is

$$y = \tfrac{1}{3}(a + x)^3 - \tfrac{1}{3}a^3.$$

3. Required the equation of a curve which passes through the point $(x' = 3, y' = 11)$, and whose slope at (x, y) is $3x^2 - 10x + 1$. $y = x^3 - 5x^2 + x + 26$.

4. The differential of an integral is $(3 + x^2)^{\frac{1}{2}} x dx$, and the integral is 0 when $x = 0$; required the value of C.

$C = -\sqrt{3}$.

5. The differential of a function is $(1 + \frac{9}{4}x)^{\frac{1}{2}} dx$, and the function is 0 when $x = 0$; find the function.

$\frac{8}{27}(1 + \frac{9}{4}x)^{\frac{3}{2}} - \frac{8}{27}$.

6. A function and its variable vanish simultaneously, and their proportional changes are as $x^2(a^2 + x^3)^{-\frac{1}{2}}$ to 1; find the function. $\frac{2}{3}(a^2 + x^3)^{\frac{1}{2}} - \frac{2}{3}a$.

Take dx for the increment of x, then $x^2(a^2 + x^3)^{-\frac{1}{2}} dx$ will be the proportional increment of the function.

7. Find the area of a plane curve whose differential is $\sqrt{4ax}\,dx$, and whose value $= 0$ when $x = 0$. $\frac{2}{3}x\sqrt{4ax}$.

8. When x is increased by dx the proportional increment of the arc of a certain curve is $(9 + x^4)^{\frac{1}{2}}x^3dx$; find the length of the arc, supposing it and x to start from the same point.

$\frac{1}{6}(9 + x^4)^{\frac{3}{2}} - 4\frac{1}{2}$.

9. The area of a surface of revolution is increased proportionally by $\pi(1 + x - x^2)^{-\frac{1}{2}}(1 - 2x)dx$ when x is increased by dx; find the area of the surface, if it $= 0$ when $x = 0$.

$[(1 + x - x^2)^{\frac{1}{2}} - 1]2\pi$.

10. The differential of the volume of a solid of revolution is $\pi(2Rx - x^2)dx$; find the volume supposing it and x to vanish simultaneously. $\pi[Rx^2 - \frac{1}{3}x^3]$.

61. Applications to Geometry. The last four problems indicate the possibility of finding the length and area of plane curves and the area of surfaces and volume of solids of revolution when their differentials are known. Now Arts. 31, 32, 33, 34 enable us to find the differentials when the equation of the curve is given.

62. Areas of Curves.

11. Find the area of $OBPA$, the equation of the curve APE being $y = x^2 - 8x + 15$, where $x = OB$ and $y = BP$.

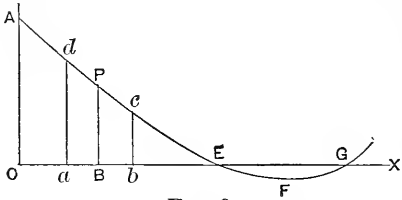

Fig. 8.

By Art. 31, the differential of $OBPA$ $(= u)$ is ydx, and in the present example $y = x^2 - 8x + 15$.

Hence, $$du = ydx = (x^2 - 8x + 15)dx;$$

$$\therefore u = \int(x^2 - 8x + 15)dx = \tfrac{1}{3}x^3 - 4x^2 + 15x + C.$$

Evidently the area $u = 0$ when $x = 0$, hence $C = 0$.

63. Definite Integrals. If, in any indefinite integral, two different values of the variable be substituted, and the one result subtracted from the other, C will be eliminated, and the result is called a definite integral.

Thus, Fig. 8, if $Oa = 1$ and $Ob = 2$, to obtain the area of the section $abcd$ we substitute 2 for x in $\frac{x^3}{3} - 4x^2 + 15x + C$ and get $16\frac{2}{3} + C$ (= area of $ObcA$), and then substitute 1 for x and obtain $11\frac{1}{3} + C$ (= area of $OadA$); subtract the latter from the former and we have $5\frac{1}{3}$ (= area of $abcd$).

The following is the notation by which these operations are indicated:

$$\int_1^2 (x^2 - 8x + 15)dx = \left[\frac{x^3}{3} - 4x^2 + 15x + C\right]_1^2 = 16\tfrac{2}{3} - 11\tfrac{1}{3} = 5\tfrac{1}{3}.$$

In general, $\int_a^b$ is the symbol of a definite integral, and indicates (1) that the differential following it is to be integrated; (2) that b and a are to be substituted successively for the variable in the indefinite integral; and (3) that the second of these results is to be subtracted from the first. The operation is called integrating between the limits a and b.

12. In the last example find the area of EFG.

Since the roots of $x^2 - 8x + 15 = 0$ are 3 and 5, $OE = 3$ and $OG = 5$. Hence we have

$$\int_3^5 ydx = \left[\frac{x^3}{3} - 4x^2 + 15x + C\right]_3^5 = -1\tfrac{1}{3} = \text{area of } EFG.$$

The result is negative because the area lies below the axis of x.

13. In the same example find the area of OEA.

$$\int_0^3 du = 18.$$

14. Find the general area of the curve $y = 3x^2 - 2x + 3$ estimated from the origin.

$$\int_0^x ydx = x^3 - x^2 + 3x.$$

15. Find the areas of the positive and negative surfaces enclosed by the curve $y = x^3 - x$ and the axis of x.

$$\int_{-1}^0 ydx = \tfrac{1}{4}; \quad \int_0^1 ydx = -\tfrac{1}{4}.$$

16. Find the entire area of $y^2 = (1 + x^2)x^2$ between the origin and the point whose abscissa is x. $\frac{2}{3}(1+x^2)^{\frac{3}{2}} - \frac{2}{3}$.

17. Find the positive area of the parabola $y^2 = 4ax$. $\frac{2}{3}xy$.

18. Find the area of $y^3 = x^6(a^3 - x^3)$ between the limits $x = 0$ and $x = a$.

$$\int_0^a (a^3 - x^3)^{\frac{1}{3}} x^2 dx = \tfrac{1}{4}a^4.$$

64. Lengths of Curves.

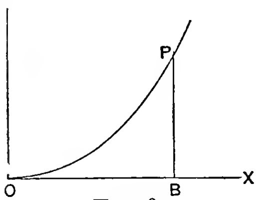

FIG. 9.

19. Find the length of the arc OP, the equation of the curve OP being $y^2 = ax^3$.

By Art. 33, the differential of OP ($= s$) is $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$. To apply this to the present curve, we differentiate $y^2 = ax^3$, and obtain $2y\,dy = 3ax^2dx$.

Hence, $$\frac{dy}{dx} = \frac{3ax^2}{2y}, \quad \left(\frac{dy}{dx}\right)^2 = \frac{9a^2x^4}{4y^2} = \frac{9ax}{4}.$$

$$\therefore \quad ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \frac{9ax}{4}}\, dx = \tfrac{1}{2}\sqrt{4 + 9ax}\, dx.$$

$$\therefore \quad s = \tfrac{1}{2}\int_0^x (4 + 9ax)^{\frac{1}{2}} dx = \frac{(4 + 9ax)^{\frac{3}{2}} - 8}{27a}.$$

20. The differential of the equation of a certain curve is $dy = \sqrt{x^3 - 1}\, dx$; find the length of the curve, beginning at the origin.

Here $$\frac{dy}{dx} = \sqrt{x^3 - 1}; \quad \therefore \left(\frac{dy}{dx}\right)^2 + 1 = x^3; \quad s = \tfrac{2}{5}x^{\frac{5}{2}}.$$

21. Find the length of the arc of a curve whose equation is $y = \frac{2}{3}(x - 1)^{\frac{3}{2}}$, measured from the point where $x = 1$. $\frac{2}{3}x^{\frac{3}{2}} - \frac{2}{3}$.

22. Find the length of a curve the differential of whose equation is $dy = \sqrt{x^2 + 2x}\, dx$, beginning at the origin.

$\frac{1}{2}x^2 + x$.

23. Find the length of the curve $y = \dfrac{x^3}{12} + \dfrac{1}{x}$, between the limits (1), $x = 2$, $x = 3$; (2), $x = a$, $x = b$.

(1) $\frac{7}{4}$; (2) $\dfrac{b^3 - a^3}{12} + \dfrac{b - a}{ab}$.

24. Find the general length of the curve $y = \left(1 - \frac{x}{3}\right)\sqrt{x}$, estimated from the origin.

$$\left(1 + \frac{x}{3}\right)\sqrt{x}.$$

65. Areas of Surfaces of Revolution.

25. AP is the arc of the circle $y^2 = R^2 - x^2$; find the area of the zone generated by revolving AP about the axis OX.

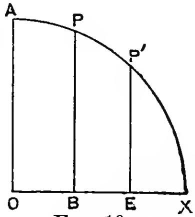

FIG. 10.

By Art. 34, $dS = 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$; in this example, since $y^2 = R^2 - x^2$, $\frac{dy}{dx} = -\frac{x}{y}$.

Hence,

$$dS = 2\pi y\sqrt{1 + \frac{x^2}{y^2}}\,dx = 2\pi R dx.$$

$$\therefore\ S = 2\pi R\int_0^x dx = 2\pi R x.$$

26. In ex. 25 find the area of the zone generated by the arc PP', supposing $OB = a$ and $OE = b$. $\left[S\right]_a^b = 2\pi R(b - a)$.

27. In ex. 25, find the surface of the generated sphere.

$$\left[S\right]_{-R}^{+R} = 4R\pi^2.$$

28. Find the surface of the cone which is generated by revolving the line $y = mx$ about the axis OX.

$$\left[S\right]_0^x = \pi y\sqrt{x^2 + y^2}.$$

29. Find the surface of the paraboloid, the generating curve being the parabola $y^2 = 4ax$.

$$\left[S\right]_0^x = \tfrac{8}{3}\pi\sqrt{a}\left[(a + x)^{\frac{3}{2}} - (a)^{\frac{3}{2}}\right].$$

30. Find the surface generated by the revolution of the curve $y = ax^3$ about the x axis.

$$\left[S\right]_0^x = \pi\frac{\left[(1 + 9a^2x^4)^{\frac{3}{2}} - 1\right]}{27a}.$$

66. Volumes of Solids of Revolution.

31. Find the volume generated by revolving $OBPA$ about the axis OX, the equation of the curve AP being $y^2 = 3x^2 - 36x + 105$.

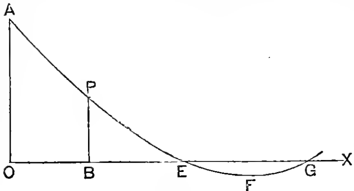

FIG. 11.

By Art. 32, $dv = \pi y^2 dx$; in this example $y^2 = 3x^2 - 36x + 105$, hence

$$dv = \pi(3x^2 - 36x + 105)dx.$$

$$\therefore v = \pi \int_0^x (3x^2 - 36x + 105)dx = \pi(x^3 - 18x^2 + 105x).$$

32. In the same example find the volume of the solid generated by the revolution (1) of OEA; (2) of EGF.

$$(1)\ [v]_0^5 = 200\pi;\quad (2)\ [v]_5^7 = -4\pi.$$

33. Find the volume of the cone which is generated by revolving about x the triangle whose base is x and whose altitude is $y\ (= mx)$.

$$v = \pi \int_0^x (m^2x^2)dx = \frac{\pi m^2 x^3}{3} = \pi y^2 \left(\frac{x}{3}\right).$$

34. Find the volume of the solid which is generated by revolving $y = x^2 - 4$ about OX.

$$v = \pi \int (x^4 - 8x^2 + 16)dx = \pi[\tfrac{1}{5}x^5 - \tfrac{8}{3}x^3 + 16x] + C.$$

35. In ex. 34, find the volume generated by the negative area of the given curve.

$$[v]_{-2}^{+2} = 34\tfrac{2}{15}\pi.$$

36. Find the volume of the spherical segments generated (1) by $OBPA$ and (2) by $BEP'P$, Fig. 10; also find (3) the volume of the sphere (see Ex. 26).

$$(1)\ [v]_0^a = \pi(R^2a - \tfrac{1}{3}a^3);\quad (2)\ [v]_a^b = \pi[R^2(b-a) - \tfrac{1}{3}(b^3 - a^3)];$$

$$(3)\ [v]_{-R}^{+R} = \tfrac{4}{3}\pi R^3.$$

37. Find the volume of a prolate spheroid, the generatrix being the ellipse $a^2y^2 = a^2b^2 - b^2x^2$.

$$[v]_{-a}^{+a} = \tfrac{4}{3}\pi ab^2.$$

CHAPTER III.

SUCCESSIVE DIFFERENTIALS AND RATE OF CHANGE.

SUCCESSIVE DIFFERENTIALS.

67. The differential of any function, as $u = f(x)$, denoted by du, is the *first differential.* The differential of the first differential, viz., $d(du)$, denoted by d^2u, is the *second differential.* The differential of the second differential, viz., $d(d^2u)$, denoted by d^3u, is the *third differential;* and in general the differential of $d^{n-1}u$, denoted by d^nu, is the *nth differential.*

du, d^2u, d^3u, etc., are the **Successive Differentials** of u.

68. If x is the independent variable, its differential (dx) is altogether independent of x,—the increment given x at any instant being entirely independent of the value which x may have at that instant. It at once follows, therefore, that the differential of dx with respect to x, like the differential of any other variable which is independent of x, is 0. Hence $d(dx) = d^2x = 0$.

For example, take the function $u = x^3$; then

$$
\begin{aligned}
&(1)\ du = 3x^2dx;\\
&(2)\ d^2u = d(3x^2)dx + 3x^2d(dx) = 6xdx^2;\\
&(3)\ d^3u = d(6x)dx^2 + 6xd(dx^2) = 6dx^3;\\
&(4)\ d^4u = d(6)dx^3 + 6d(dx^3) = 0.
\end{aligned}
$$

Therefore, in finding the successive differentials of a function, we treat the differential of the independent variable as a constant. See Appendix, A_3.

The student should observe the difference between expressions like d^2y, dy^2, and $d(y^2)$; d^2y is the second differential of y, dy^2 is the square of the differential of y, and $d(y^2)$ is the differential of y^2, or $2ydy$.

If $u = f(x)$, the *successive derivatives* or *differential coefficients* of u with respect to x are $\frac{du}{dx}, \frac{d^2u}{dx^2}, \frac{d^3u}{dx^3}$, etc.

The successive derivatives of $f(x)$ are also denoted by $f'(x)$, $f''(x), f'''(x), \ldots f^n(x)$.

Therefore, when $u = f(x)$, we have

$$\frac{du}{dx} = f'(x), \quad \frac{d^2u}{dx^2} = f''(x), \quad \ldots \quad \frac{d^nu}{dx^n} = f^n(x).$$

EXAMPLES.

1. Find the successive differentials of $y = x^4$.

$$\begin{aligned} dy &= 4x^3dx. \\ d^2y &= d(4x^3dx) = 4dxd(x^3) = 12x^2dx^2. \\ d^3y &= d(12x^2dx^2) = 12dx^2d(x^2) = 24xdx^3. \\ d^4y &= d(24xdx^3) = 24dx^3d(x) = 24dx^4. \\ d^5y &= d(24dx^4) = 0. \end{aligned}$$

2. Find the successive derivatives of $x^3 - 4x^2 + 3x - 5$.

Let $f(x) = x^3 - 4x^2 + 3x - 5$;

then
$$\begin{aligned} f'(x) &= \frac{d}{dx}(x^3 - 4x^2 + 3x - 5) = 3x^2 - 8x + 3; \\ f''(x) &= \frac{d}{dx}(3x^2 - 8x + 3) = 6x - 8; \\ f'''(x) &= \frac{d}{dx}(6x - 8) = 6; \\ f^{\text{iv}}(x) &= 0. \end{aligned}$$

3. Find the successive differentials of $u = 2x^3 - 7x^2$.

$du = (6x^2 - 14x)dx$; $d^2u = (12x - 14)dx^2$; $d^3u = 12dx^3$.

4. $u = x^4 - 3x^3 + 5x$.

5. $u = x^{-3}$.

$du = -3x^{-4}dx$; $d^2u = 12x^{-5}dx^2$; $d^3u = -60x^{-6}dx^3$; etc.

6. $u = x^{-2} - x^{-1} + 5$.

$du = -(2x^{-3} - x^{-2})dx$; $d^2u = (6x^{-4} - 2x^{-3})dx$; etc.

7. $u = (x + 1)^{\frac{1}{3}}$.

$du = \frac{1}{3}(x+1)^{-\frac{2}{3}}dx$; $d^2u = -\frac{2}{9}(x + 1)^{-\frac{5}{3}}dx^2$; etc.

8. If $f(x) = \frac{1}{x + 2}$, show that $f'''(x) = -\frac{6}{(x + 2)^4}$.

9. If $f(x) = \dfrac{1}{3x+4}$, show that $f^{\text{iv}}(1) = \dfrac{\underline{|4}* \, 3^4}{7^5}$.

10. If $f(x) = \dfrac{1-x}{1+x}$, show that $f''(1) = \dfrac{1}{2}$.

11. If $f(x) = \dfrac{3x+2}{x^2-4}$, show that $f'''(0) = -\dfrac{9}{8}$.

69. Leibnitz's Theorem. To find the successive differentials of the product of two variables.

Let u and v be functions of x; then

$$d(uv) = udv + vdu. \quad \dots\dots \quad (1)$$

Differentiating (1), regarding u, v, du, dv, as functions of x, we have

$$d^2(uv) = ud^2v + dudv + dvdu + vd^2u,$$

or

$$d^2(uv) = ud^2v + 2dudv + vd^2u. \quad \dots\dots \quad (2)$$

Differentiating (2), we get

$$d^3(uv) = ud^3v + 3dud^2v + 3d^2udv + vd^3u,$$

from which we see that the law of the coefficients is evidently the same as in the binomial formula.

RATE OF CHANGE.

70. Uniform Change. When a variable changes uniformly it experiences equal changes in equal intervals of time, whatever the magnitude of these intervals.

Thus, we may suppose the line ak $(= x)$ to be generated by a point moving over each of the equal distances ab, bc, cd, etc.,

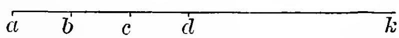

FIG. 12.

in a second of time. If $ab = bc = cd \ldots = h$ in., *the rate of change of* x is h in. per second. Denoting the number of seconds

* $\underline{|4}$, called factorial 4, means $1 \times 2 \times 3 \times 4$. In general, $\underline{|n} = 1 \times 2 \times 3 \ldots \times n$.

by t, we have $x = ht$. Differentiating this equation, regarding h as constant, we have

$$dx = hdt, \quad \text{and} \quad \frac{dx}{dt} = h.$$

Therefore, I. *In uniform change the increment of the variable varies as the increment of the time.*

II. *The derivative* $\frac{dx}{dt}$ *expresses the rate of change of x per second, when t represents seconds.*

71. Variable Change. When a variable does not change uniformly, its rate of change is ever changing, and is measured at any instant by what the change or increment would be during the next unit of time if it (the rate of change) were to remain unchanged during that time.

Thus, when a train leaves a station it usually goes, for some distance, faster and faster; that is, its rate of change or velocity is constantly accelerated. If at any instant during this time we say "It is going at the rate of 20 miles per hour," we mean that it would travel that distance the next hour if its present rate should remain unchanged, or, as in uniform change, if the increment of the distance should vary as the increment of the time, that is, uniformly.

The same is true of any other variable; that is, u being a function of time (t), the rate of change of u is measured by what Δu would be in the next interval of time if it varied as Δt or dt; but du is what Δu would be if it so varied, Art. 25, hence the differential of u measures its rate of change.

Cor. I. Since $du \propto dt$, $du = mdt$,

or
$$\frac{du}{dt} = m,$$

where $m \left(= \frac{du}{dt}\right)$ is the rate of change of u per second, and, being a function of t, is in general variable. See Appendix, A_2.

EXAMPLES.

1. Required the rate of change of the area of a square.

Let $x =$ the side and u the area; then

$$u = x^2, \quad \text{and} \quad du = 2x dx. \quad \therefore \frac{du}{dt} = 2x\frac{dx}{dt}.$$

That is, the rate of change of the area $\left(\frac{du}{dt}\right)$ is equal to the rate of change of the side $\left(\frac{dx}{dt}\right)$ multiplied by twice the side ($2x$).

We may omit dt and regard du and dx as the measures of the rates of change of u and x. Thus, $du = 2xdx$ signifies that the area is increasing in square inches $2x$ times as fast as the side is increasing in linear inches.

2. In the function $u = x^2 - 4x + 5$, (1) at what rate is u increasing when x is 5 in. and increasing at the rate of 3 in. per second? (2) At what rate is x increasing when it is 10 in. and u is increasing at the rate of 40 in. per second? (3) What is the value of x at the point where u is increasing 10 times as fast as x?

Differentiating the given function, we have

$$du = (2x - 4)dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In this equation we have three quantities, viz.: du, the rate of increase of u; dx, the rate of increase of x; and x; hence, when either two of these are given the third may be found.

(1) $x = 5$ and $dx = 3$; substituting in (1) and we have

$$du = (10 - 4)3 = 18, \text{ Ans.}$$

(2) $du = 40$, $x = 10$; substituting in (1) and we have

$$40 = (20 - 4)dx, \text{ whence } dx = 2\tfrac{1}{2}, \text{ Ans.}$$

(3) $du = 10dx$; substituting in (1), we have

$$10dx = (2x - 4)dx, \quad \text{or} \quad 10 = 2x - 4, \quad \text{whence } x = 7, \text{ Ans.}$$

3. If x increases uniformly at the rate of 5 inches per second, at what rate is $u = x^3 - 4x$ increasing when $x = 10$ inches? 1480 in. per second.

4. In the function $y = 2x^3 + 6$, what is the value of x at the point where y increases 24 times as fast as x? $x = \pm 2$.

5. If the side of a square increases uniformly at the rate of 3 inches per second, what is the length of the side at the time the area is increasing at the rate of 20 sq. inches per second?

Let $x =$ the side, $u =$ the area; then $u = x^2$.

6. In the last example, supposing the area to increase uniformly at the rate of 10 sq. inches per second, at what rate will the side be increasing when the area is 22 sq. inches?

Take $x = \sqrt{u}$. $\frac{5}{\sqrt{22}}$ in. per second.

7. A circular plate of metal expands by heat so that its diameter increases uniformly at the rate of 2 inches per second; at what rate is the surface increasing when the diameter is 5 inches?

Let $x =$ the diameter, $u =$ the area; then $u = \frac{\pi}{4}x^2$.

5π sq. in. per second.

8. In the last problem, if the surface increases uniformly at the rate of 50 sq. inches per second, at what rate will the diameter be increasing when it becomes 5 inches?

$\frac{20}{\pi}$ in. per second.

9. The volume of a spherical soap-bubble increases how many times as fast as the radius? When its radius is 4 in., and increasing at the rate of $\frac{1}{2}$ in. per second, how fast is the volume increasing?

Let $x =$ the radius, $u =$ volume; then $u = \frac{4}{3}\pi x^3$.

(1) $4\pi x^2$ times as fast. (2) 32π cu. in. per second.

10. A ladder 50 ft. long is leaning against a perpendicular wall, the foot of the ladder being on a horizontal plane x ft. from the base of the wall. Suppose the foot of the ladder to be pulled away from the wall at the rate of 3 ft. per minute. (1) How fast is the top of the ladder descending when $x = 14$ ft.? (2) How fast is it descending when $x = 30$ ft? (3) What is the value of x when the top of the ladder is descending at the rate

of 4 ft. per minute? (4) And what at the time the bottom and top of the ladder are moving at the same rate?

Let $y =$ the distance from the base of the wall to the top of the ladder; then $y = \sqrt{2500 - x^2}$.

(1) $\frac{7}{8}$ ft. per minute. (2) $2\frac{1}{4}$ ft. per minute. (3) $x = 40$ ft. (4) $x = 25\sqrt{2}$ ft.

11. What is the value of x at the point where $x^3 - 5x^2 + 17x$ and $x^3 - 3x$ change at the same rate? $x = 2$.

12. Find the values of x at the points where the rate of change of $x^3 - 12x^2 + 45x - 13$ is zero. $x = 3$, $x = 5$.

13. In a parabola whose equation is $y^2 = 12x$, if x increases uniformly at the rate of 2 in. per second, at what rate is y increasing when $x = 3$ inches? 2 in. per second.

14. In the same parabola, at what point do y and x vary at the same rate? When $y = 6$.

15. In the ellipse whose equation is $44\frac{4}{9}y^2 + 25x^2 = 1111\frac{1}{9}$, at what point of the curve does y decrease at the same rate that x increases? When $y = 3$ and $x = 5\frac{1}{3}$.

16. Find the points where the rate of change of the ordinate $y = x^3 - 6x^2 + 3x + 5$ is equal to the rate of change of the slope of the curve. Where $x = 1$ and $x = 5$.

17. Two straight roads intersect at right angles; a bicyclist travelling the one at the rate of 10 miles per hour passes the intersection $2\frac{1}{2}$ hours in advance of another travelling the other road at the rate of 8 miles per hour. At what rate were they separating (1) at the end of $1\frac{1}{2}$ hours after the first man passed the intersection? (2) At the end of $2\frac{1}{2}$ hours? (3) Required the distance (y) between them when it is not changing.

$$(1)\ \frac{dy}{dt} = 5\tfrac{1}{17};\quad (2)\ \frac{dy}{dt} = 10;\quad (3)\ y = \tfrac{100}{41}\sqrt{41}\ \text{mi.}$$

72. Applications to Geometry. The rates of change of the areas and lengths of curves and of the areas and volumes of surfaces and solids of revolution are given by Arts. 31, 32, 33, 34. The applications are made as in Arts. 62, 64, 65, 66.

Rate of Change of Curves. Formula, $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$

18. Find the rate of change of the arc of the parabola $y^2 = 4ax.$ $\qquad ds = \sqrt{1 + \frac{a}{x}}\,dx.$

19. In the previous example, if $a = 9$, and x increases at the rate of 12 inches per second, at what rate will the arc be increasing when $x = 16$ inches? $\qquad ds = 15$ in. per second.

20. Show that the rate of change of the arc of the circle $x^2 + y^2 = R^2$ is $ds = \dfrac{Rdx}{\sqrt{R^2 - x^2}}.$

21. In a circle whose radius is 20 in., the abscissa changes at the rate of n in. per sec.; at what rate is the arc increasing when $x = 12$ in.? $\qquad \frac{5}{4}n$ in. per second.

73. Given the rate of change of the arc of a curve (ds), to find the rates of change of its co-ordinates x and y.

Take the parabola $y^2 = 4ax$, then $ydy = 2adx$; between this equation and $ds^2 = dx^2 + dy^2$ eliminate (1) dx and (2) dy, and we have

$$dy = \frac{2a}{\sqrt{4a^2 + y^2}}\,ds \quad \text{and} \quad dx = \sqrt{\frac{x}{a + x}}\,ds.$$

22. In the parabola $y^2 = 4x$, if the arc increases uniformly at the rate of 5 inches per second, at what rates are y and x increasing when $x = 9$ inches?

$\frac{1}{2}\sqrt{10}$ and $\frac{3}{2}\sqrt{10}$ inches per second.

23. If the arc of the circle $x^2 + y^2 = 100$ increases at the rate of 5 inches per second, at what rates are y and x changing when $x = 6$ inches? $\qquad -3$ and $+4$ in. per second.

APPLICATION TO MECHANICS.

74. Velocity is the rate of change of the distance described by a moving body. Hence, if $s =$ the distance, $v =$ the velocity and $t =$ the time, we have

$$\text{I.} \qquad v = \frac{ds}{dt}; \quad \therefore s = \int vdt, \quad \text{and} \quad t = \int \frac{ds}{v}.$$

Again, denoting the rate of change of the velocity by a', we have

$$\text{II.} \qquad a' = \frac{dv}{dt} = \frac{d^2s}{dt^2}; \quad \therefore v = \int a'dt, \quad \text{and} \quad t = \int \frac{dv}{a'}.$$

EXAMPLES.

1. If $s = 3t^2$, what is the velocity and its rate of change?

Since $s = 3t^2$, $\qquad \dfrac{ds}{dt} = 6t = v$, the velocity;

and since $v = 6t$, $\qquad \dfrac{dv}{dt} = 6 = a'$, the rate of change of v.

Thus, if the unit of s is one foot, and the unit of t one second, $v = \dfrac{ds}{dt}$ $[= 6t]$ ft. per second, and $a' = \dfrac{dv}{dt}$ $[= 6]$ ft. per second.

2. A body passes over a distance of $ct^{\frac{1}{2}}$ in t seconds; find v and a', (1) in general, and (2) at the end of 9 seconds.

(1) $\dfrac{c}{2\sqrt{t}}$ and $-\dfrac{c}{4\sqrt{t^3}}$; (2) $\dfrac{c}{6}$ and $-\dfrac{c}{108}$.

3. A body after moving t seconds has a velocity of $3t^2 + 2t$ ft. per second; find its distance from the point of starting.

$$s = \int vdt = t^3 + t^2.$$

4. The velocity of a body after moving t seconds is $5t^2$ ft. per second; (1) how far will it be from the point of starting in 3 seconds? (2) In what time will it pass over a distance of 360 feet?

(1) 45 ft.; (2) 6 sec.

5. A body moves from A, and in t seconds its velocity is $14t$ ft. per second; (1) how far is the body from A? (2) In how many seconds will the body have gone 847 feet?

(1) $7t^2$ ft.; (2) 11 seconds.

75. The velocity is positive or negative according as s is increasing or decreasing, and a' sustains the same relation to v;

therefore, if s increases as the moving body goes *forward* and decreases as it goes *backward*, the body is moving forward or backward according as v is *positive* or *negative*.

6. A train left a station and in t hours was at a distance of $\frac{1}{4}t^4 - 4t^3 + 16t^2$ miles from the starting-point; required the velocity and its rate of change, also when the train was moving backward, when the velocity or rate per hour was decreasing, and the entire distance travelled in 10 hours.

$$s = \tfrac{1}{4}t^4 - 4t^3 + 16t^2 = \text{the distance from station.}$$

$$\frac{ds}{dt} = t^3 - 12t^2 + 32t = v = \text{the velocity.}$$

$$\frac{dv}{dt} = 3t^2 - 24t + 32 = a' = \text{the rate of change of } v.$$

The roots of $t^3 - 12t^2 + 32t = 0$ are 0, 4 and 8; therefore v is negative, and the train was moving backward from the 4th to the 8th hour.

Again, the roots of $3t^2 - 24t + 32 = 0$ are $1.7-$ and $6.3+$; hence a' is negative, and therefore v was decreasing, from the 1.7th to the 6.3th hour.

The roots of $\frac{1}{4}t^4 - 4t^3 + 16t^2 = 0$ are 0, 0, 8 and 8; that is, $s = 0$ when $t = 8$; hence the train was at the starting-point at the end of 8 hours, having gone backward as far as it had forward.

Since the train moved forward the first four hours, then backward the next four hours, and then forward, the entire distance passed over in 10 hours was

$$[s]_0^4 - [s]_4^8 + [s]_8^{10} = 64 + 64 + 100 = 228 \text{ (miles).}$$

7. A train left a station and in t hours was moving at the rate of $t^3 - 21t^2 + 80t$ miles per hour; required (1) the distance from the starting-point; (2) when the train was moving backward; (3) when its rate per hour was decreasing; (4) when the train repassed the station; and (5) how far it had travelled when it passed the starting-point the last time.

(1) $s = \int vdt = \int (t^3 - 21t^2 + 80t)dt = \frac{1}{4}t^4 - 7t^3 + 40t^2.$

(2) From the 5th to the 16th hour; (3) from the 2.27th to the 11.72th hour; (4) in 8 and 20 hours;

$$(5) \quad \Big[s\Big]_0^5 - \Big[s\Big]_5^{16} + \Big[s\Big]_{16}^{20} = 4658\tfrac{1}{2} \text{ miles.}$$

8. A traveller left a point A at 12 M., and in t hours after his rate per hour was $5 - t$ miles; (1) how far forward did he go? (2) At what times was he 8 miles from A? (3) What were his rates per hour when at a distance of $10\frac{1}{2}$ miles from A?

(1) $12\frac{1}{2}$ miles; (2) 2 P.M. and 8 P.M.; (3) $+2$ mi. and -2 mi. per hour.

76. Uniformly Accelerated Motion is that in which the rate of change of the velocity (a') is constant. That is, v changes uniformly or $v \propto t$; hence, Art. 70, $\frac{\Delta v}{\Delta t} = \frac{dv}{dt} = a' =$ the rate of change of the velocity.

Formulas. $$v = \int a'dt = a't + C = a't + v_0, \quad . \quad . \quad (1)$$

and $$s = \int vdt = \int (a't + v_0)dt = \tfrac{1}{2}a't^2 + v_0 t + s_0, \quad . \quad (2)$$

in which v_0 and s_0 represent the initial velocity and distance; that is, the values of v and s when $t = 0$.

If $v_0 = 0$ and $s_0 = 0$ when $t = 0$, then (1) and (2) become

$$v = a't, \quad s = \tfrac{1}{2}a't^2; \quad \therefore t = \sqrt{\frac{2s}{a'}} \text{ and } v = \sqrt{2a's}. \quad . \quad (3)$$

77. The increment of v, or acceleration, produced by gravity is about 32.17 ft. per second, and is usually represented by g. Hence, by substituting g for a' in (3), we obtain the four formulas for *the free fall of bodies in vacuo* near the earth's surface. When the bodies are not in vacuo the formulas generally are slightly inaccurate, on account of the resistance of the atmosphere.

PROBLEMS.

1. A rifle-ball is projected from O in the direction of Y with a velocity of b ft. per second; required its path, knowing that its velocity in t seconds along the action-line of gravity (OX) will be gt ft. per second.

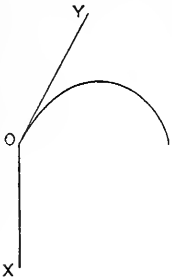

FIG. 13.

Let OX and OY be the axis of x and y, respectively; then $\frac{dy}{dt} = b$, and $\frac{dx}{dt} = gt$;

$$\therefore \quad y = bt, \ldots (1) \quad \text{and} \quad x = \tfrac{1}{2}gt^2. \ldots (2)$$

Eliminating t between (1) and (2), we have

$$y^2 = \frac{2b^2}{g}x,$$

that is, the path of the ball is an arc of a parabola.

2. A body starts from O, and in t seconds its velocity in the direction of OX is $2abt$, and in the direction of OY is $a^2t^2 - b^2$; find its velocity along its path Onm, the distances in the direction of each axis and along the line of its path, and the equation of its path, the axes being rectangular.

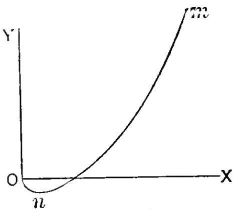

FIG. 14.

Let v_x, v_y and v_s represent respectively the velocities in the directions of the axes x and y and the path s. Then

$$v_x = \frac{dx}{dt} = 2abt; \qquad \therefore x = \int 2abt\,dt = abt^2; \quad \ldots \quad (1)$$

$$v_y = \frac{dy}{dt} = a^2t^2 - b^2; \quad \therefore y = \int (a^2t^2 - b^2)dt = \tfrac{1}{3}a^2t^3 - b^2t; \quad (2)$$

$$v_s = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{a^4t^4 + 2a^2b^2t^2 + b^4} = a^2t^2 + b^2.$$

$$\therefore \quad s = \int (a^2t^2 + b^2)dt = \tfrac{1}{3}a^2t^3 + b^2t. \quad \ldots \ldots \quad (3)$$

Now to find the path of the body, we eliminate t between (1) and (2) and obtain

$$y = \left(\frac{ax}{3b} - b^2\right)\sqrt{\frac{x}{ab}}.$$

CHAPTER IV.

GENERAL DIFFERENTIATION.

LOGARITHMS.

78. LEMMA. *The limit of* $\left(1+\frac{1}{z}\right)^z$, *as* z *approaches infinity, is the sum of the infinite series* $2+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+$ *etc.*

Assuming the binomial theorem for positive integral values of z, we have

$$\left(1+\frac{1}{z}\right)^z = 1 + z\frac{1}{z} + \frac{z(z-1)}{\underline{|2}}\frac{1}{z^2} + \frac{z(z-1)(z-2)}{\underline{|3}}\frac{1}{z^3} + \text{etc.}$$

$$= 1 + 1 + \frac{1}{\underline{|2}}\left(1-\frac{1}{z}\right) + \frac{1}{\underline{|3}}\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right) + \text{etc.},$$

which evidently approaches $2+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+$ etc., as z approaches infinity. The sum of this series is represented by e. It is the base of the natural system of logarithms, which is equal to 2.718281, approximately.

$$\therefore \qquad \lim_{z \doteq \infty}\left(1+\frac{1}{z}\right)^z = e.$$

79. To differentiate the logarithm of a variable.

Let $u = \log_a v$, where v is a function of x.

When x is increased by h we have

$$\begin{aligned} \Delta u &= \log_a(v + \Delta v) - \log_a v \\ &= \log_a\left(\frac{v + \Delta v}{v}\right) \\ &= \frac{\Delta v}{v}\log_a\left(1 + \frac{\Delta v}{v}\right)^{\frac{v}{\Delta v}}. \end{aligned}$$

$$\therefore \qquad \frac{\Delta u}{\Delta v} = \frac{1}{v}\log_a\left(1 + \frac{\Delta v}{v}\right)^{\frac{v}{\Delta v}}.$$

Passing to the limit, remembering that as h approaches 0, $\frac{\Delta v}{v}$ approaches 0, and $\frac{v}{\Delta v}$ approaches ∞, we have, as in the preceding lemma,

$$\frac{du}{dv} = \frac{1}{v}\log_a e, \quad \text{or} \quad du = (\log_a e)\frac{dv}{v}.\text{*}$$

RULE.—*Divide the differential of the variable by the variable itself, and multiply the quotient by the constant* $\log_a e$.

The factor $\log_a e$, usually represented by m, is called the modulus of the system whose base is a. When $a = e$ the modulus is unity and we have $du = \frac{dv}{v}$, simply. Herein lies the advantage of the natural system of logarithms, whose base is e, in all discussions of a theoretical nature. Hereafter when the base of a given logarithm is not indicated, it is to be understood that the base is e.

EXAMPLES.

1. Differentiate $u = \log(x^3 - 2x + 5)$.

$$du = \frac{d(x^3 - 2x + 5)}{x^3 - 2x + 5} = \frac{(3x^2 - 2)dx}{x^3 - 2x + 5}.$$

---

* For another method, see Appendix, A_4, Cor. III.

2. $u = \log\left(\frac{1+x}{1-x}\right)$.

$$du = d\left(\frac{1+x}{1-x}\right) \div \frac{1+x}{1-x} = \frac{2dx}{(1-x)^2} \times \frac{1-x}{1+x} = \frac{2dx}{1-x^2}.$$

3. $u = \log \sqrt{x^3 - a^3}$. $\qquad du = \frac{3x^2dx}{2(x^3 - a^3)}$.

4. $u = \log_a (5x^2 - x^3)^4$. $\qquad du = \frac{4m(10 - 3x)dx}{5x - x^2}$.

5. $u = \log (x + \sqrt{1 + x^2})$. $\qquad du = \frac{dx}{\sqrt{1 + x^2}}$.

6. $u = \log \left[(a - x) \sqrt{a + x}\right]$. $\qquad du = -\frac{(a + 3x)dx}{2(a^2 - x^2)}$.

7. $y = \log^3 x$. $\qquad dy = 3 \log^2 x \frac{dx}{x}$.

8. $y = \log^4 (\log x)$. $\qquad dy = \frac{4 \log^3 (\log x)dx}{x \log x}$.

9. $y = \log (\log^4 x)$. $\qquad dy = \frac{4dx}{x \log x}$.

10. $y = \log (x + a + \sqrt{2ax + x^2})$. $\qquad dy = \frac{dx}{\sqrt{2ax + x^2}}$.

EXPONENTIAL FUNCTIONS.

80. To differentiate $u = a^v$.

Passing to logarithms, we have

$$\log u = v \log a.$$

Differentiating, $\qquad \frac{du}{u} = dv \log a.$

Multiplying by u, $\qquad du = u \log a \, dv.$

Substituting a^v for u, $\qquad du = a^v \log a \, dv.$*

---

* For another method, see Appendix, A_4.

Hence, *the differential of an exponential function with a constant base is equal to the function itself into the natural logarithm of the base into the differential of the exponent.*

Cor. I. $d(e^v) = e^v dv$, since $\log e = 1$.

EXAMPLES.

1. Differentiate $y = m^{\sqrt{1+x^2}}$.

$$dy = m^{\sqrt{1+x^2}} \log(m) d(\sqrt{1+x^2}) = \frac{m^{\sqrt{1+x^2}} \log(m) x\, dx}{\sqrt{1+x^2}}.$$

2. $y = e^{\log x}$. $\qquad dy = \dfrac{e^{\log x} dx}{x}$.

3. $y = e^{x \log x}$. $\qquad dy = e^{x \log x}(\log x + 1)dx$.

4. $y = \dfrac{e^x}{1+x}$. $\qquad dy = \dfrac{e^x x\, dx}{(1+x)^2}$.

5. $y = e^x(1 - x^3)$. $\qquad dy = e^x(1 - 3x^2 - x^3)dx$.

6. $y = \log \dfrac{e^x}{1+e^x}$. $\qquad dy = \dfrac{dx}{1+e^x}$.

7. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. $\qquad dy = \left(\dfrac{2}{e^x + e^{-x}}\right)^2 dx$.

81. To differentiate $u = y^v$.

Passing to logarithms, we have

$$\log u = v \log y.$$

Differentiating, $\qquad \dfrac{du}{u} = dv \log y + v\dfrac{dy}{y}$.

Multiplying by u, $\qquad du = u \log y dv + uv\dfrac{dy}{y}$.

Substituting y^v for u, $\qquad du = y^v \log y dv + vy^{v-1}dy$.

Hence, *the differential of an exponential function with a variable base is the sum of the results obtained by first differentiating as though the base were constant, and then as though the exponent were constant.*

The method of differentiating a function by first passing to logarithms, as in the two preceding demonstrations, is called *logarithmic differentiation.* It may be used to great advantage in differentiating many exponential functions and those involving products and quotients.

EXAMPLES.

1. Differentiate $y = (x^2 + 1)^{x+1}$.

$$dy = (x^2 + 1)^{x+1} \log (x^2 + 1)d(x + 1) + (x + 1)(x^2 + 1)^x d(x^2 + 1)$$
$$= (x^2 + 1)^x[(x^2 + 1) \log (x^2 + 1) + 2(x^2 + x)]dx.$$

2. $y = x^x$. $\qquad dy = x^x(\log x + 1)dx$.

3. $y = \sqrt[x]{x}$ or $x^{\frac{1}{x}}$. $\qquad dy = \dfrac{\sqrt[x]{x}\,(1 - \log x)dx}{x^2}$.

4. $y = \dfrac{e^x - 1}{e^x + 1}$. $\qquad dy = \dfrac{2e^x}{(e^x + 1)^2}dx$.

5. $y = e^{x^x}$. $\qquad dy = e^{x^x}(1 + \log x)x^x dx$.

Make $u = x^x$, differentiate, and replace u and du by their values.

6. $y = x^{x^x}$. $\qquad dy = x^{x^x}\left(\dfrac{1}{x} + \log x + \log^2 x\right)x^x dx$.

7. $y = \left(\dfrac{a}{x}\right)^x$. $\qquad dy = \left(\dfrac{a}{x}\right)^x\left(\log \dfrac{a}{x} - 1\right)dx$.

TRIGONOMETRIC FUNCTIONS.

82. Circular Measure of Angles. If $v =$ the length of the circular arc BP, and $r =$ the length of the radius OB in terms of the same unit, the ratio $\dfrac{v}{r}$ is the circular measure of the angle BOP. When $r = 1$, the measure of the angle is simply v, which is the length of the arc. This method of measuring an angle is called the *circular* or *analytic* system, as distinguished from the *degree* or *gradual* method.

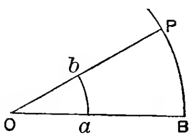

FIG. 15.

83. To differentiate $\sin v$ **and** $\cos v$.

With the radius OB $(=1)$ describe the circle whose centre is O, and let the angle BOP or its measuring arc BP be any value of v, which is a function of x, then $PE = \sin v$ and $OE = \cos v$.

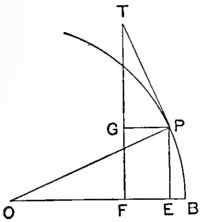

FIG. 16.

Let us suppose BP or v to receive an increment. Take PT, a part of the tangent line at P, for the differential (dv) of the arc BP (Art. 48), then the proportional increments of $\sin v$ and $\cos v$ will be GT and $-GP$, respectively. Therefore $GT = d(\sin v)$, and $-GP = d(\cos v)$.

The angle $GTP = BOP = v$; hence in the triangle GTP we have

(1) $\quad GT = \cos v\, dv, \qquad \therefore\ d(\sin v) = \cos v\, dv.$

(2) $\quad GP = \sin v\, dv, \qquad \therefore\ d(\cos v) = -\sin v\, dv.$

Hence, *the differential of the sine of an angle is the cosine of the angle into the differential of the angle.*

The differential of the cosine of an angle is minus the sine of the angle into the differential of the angle.

84. COR. I. $TF = \sin(v + dv)$ and $OF = \cos(v + dv)$, which approach, respectively, PT and OB as v diminishes, and when $v = 0$ we have $\sin dv = dv$ and $\cos dv = 1$. That is, *the sine of the differential of an arc is the differential of the arc itself, and the cosine of the differential of an arc is* 1.

85. *The differential of the tangent of an angle is equal to the square of the secant of the angle into the differential of the angle.*

For, $\qquad \tan v = \dfrac{\sin v}{\cos v}.$

$$\therefore \qquad d \tan v = \frac{\cos v d \sin v - \sin v d \cos v}{\cos^2 v} \qquad \text{(Art. 41)}$$

$$= \frac{(\cos^2 v + \sin^2 v)dv}{\cos^2 v} = \sec^2 v\, dv.$$

86. *The differential of the cotangent of an angle is equal to minus the square of the cosecant of the angle into the differential of the angle.*

For, $\cot v = \dfrac{1}{\tan v}.$

$$\therefore \quad d \cot v = -\frac{d \tan v}{\tan^2 v} = -\frac{\sec^2 v dv}{\tan^2 v} = -\operatorname{cosec}^2 v\, dv.$$

87. *The differential of the secant of an angle is equal to the secant of the angle into the tangent of the angle into the differential of the angle.*

For, $\sec v = \dfrac{1}{\cos v};$

$$\therefore \quad d \sec v = -\frac{d \cos v}{\cos^2 v} = \frac{\sin v\, dv}{\cos^2 v} = \sec v \tan v\, dv.$$

88. *The differential of the cosecant of an angle is equal to minus the cosecant of the angle into the cotangent of the angle into the differential of the angle.*

For, $\operatorname{cosec} v = \dfrac{1}{\sin v};$

$$\therefore \quad d \operatorname{cosec} v = -\frac{d \sin v}{\sin^2 v} = -\frac{\cos v\, dv}{\sin^2 v} = -\operatorname{cosec} v \cot v\, dv.$$

89. $d \operatorname{vers} v = d(1 - \cos v) = \sin v\, dv.$

EXAMPLES.

Differentiate the following:

1. $y = \sin(x^2 - x).$

$$dy = \cos(x^2 - x) d(x^2 - x) = \cos(x^2 - x)(2x - 1)dx.$$

2. $y = \tan^4 (x^3).$

$$\begin{aligned} dy &= 4 \tan^3 (x^3) d(\tan x^3) = 4 \tan^3 (x^3) \sec^2 (x^3) d(x^3) \\ &= 12 \tan^3 (x^3) \sec^2 (x^3) x^2 dx. \end{aligned}$$

3. $y = \cos(ax)$. $\qquad dy = -a \sin(ax)dx$.
4. $y = \cos^3 x$. $\qquad dy = -3 \cos^2 x \sin x\, dx$.
5. $y = \sin x \cos x$. $\qquad dy = (\cos^2 x - \sin^2 x)dx$.
6. $y = \tan^2 5x$. $\qquad dy = 10 \tan(5x) \sec^2(5x)dx$.
7. $y = \sec^3 x$. $\qquad dy = 3 \sec^3 x \tan x\, dx$.
8. $y = \sin x \tan x$. $\qquad dy = (\sin x + \tan x \sec x)dx$.
9. $y = (x \cot x)^2$. $\qquad dy = 2x \cot x(\cot x - x \operatorname{cosec}^2 x)dx$.
10. $y = x \sin x + \cos x$. $\qquad dy = x \cos x\, dx$.
11. $y = x - \sin x \cos x$. $\qquad dy = 2 \sin^2 x\, dx$.
12. $y = \tan x - x$. $\qquad dy = \tan^2 x\, dx$.
13. $y = \sin(\cos x)$. $\qquad dy = -\sin x \cos(\cos x)dx$.
14. $y = \sin x - \frac{1}{3} \sin^3 x$. $\qquad dy = \cos^3 x\, dx$.
15. $y = \frac{1}{3} \cos^3 x - \cos x$. $\qquad dy = \sin^3 x\, dx$.
16. $y = \frac{1}{3} \tan^3 x - \tan x + x$. $\qquad dy = \tan^4 x\, dx$.
17. $y = x^{\sin x}$. $\qquad dy = x^{\sin x}\left[\frac{\sin x}{x} + \log x \cos x\right]dx$.
18. $y = (\sin x)^x$. $\qquad dy = (\sin x)^x[\log \sin x + x \cot x]dx$.

Prove the following by differentiating both members (see Art. 50):

19. $\int \cot x\, dx = \log(\sin x) + C$.

20. $\int -\tan x\, dx = \log(\cos x) + C$.

21. $\int \sec x \operatorname{cosec} x\, dx = \log(\tan x) + C$.

22. $\int -\sec x \operatorname{cosec} x\, dx = \log(\cot x) + C$.

23. $\int \tan x\, dx = \log(\sec x) + C$.

24. $\int -\cot x\, dx = \log(\operatorname{cosec} x) + C$.

25. $\int \frac{\sin x\, dx}{1 - \cos x} = \log(\operatorname{vers} x) + C$.

26. $\int \sin x \cos^3 x\, dx = \frac{1}{2} \sin^2 x - \frac{1}{4} \sin^4 x + C$.

27. $\int \sin^3 x \cos^3 x\, dx = \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + C$.

28. $\int \frac{\sin^3 x}{\cos^2 x} dx = \sec x + \cos x + C.$

Prove the following:

29. If $f(x) = (x^2 - 6x + 12)e^x, f'''(x) = x^2 e^x.$
30. If $f(x) = e^{-x} \cos x, f^{\text{iv}}(x) = -4e^{-x} \cos x.$
31. If $f(x) = \tan x, f'''(x) = 6 \sec^4 x - 4 \sec^2 x.$
32. If $y = \frac{7 \cos x}{9} - \frac{\cos^3 x}{27}, \frac{d^3 y}{dx^3} = \sin^3 x.$
33. If $f(x) = x^3 \log x, f^{\text{iv}}(1) = 6.$
34. If $f(x) = \log \sin x, f'''\left(\frac{\pi}{4}\right) = 4.$
35. If $f(x) = \sin x, f'(0) = 1; f'''(0) = -1.$
36. If $f(x) = \log (1 + x), f''(0) = -1; f^{\text{v}}(0) = \underline{|4}.$
37. If $f(x) = a^x, f^n(0) = \log^n a.$
38. If $f(x) = e^x \log x, f^{\text{iv}}(c) = e^c\left[\log c + \frac{4}{c} - \frac{6}{c^2} + \frac{8}{c^3} - \frac{6}{c^4}\right].$

INVERSE TRIGONOMETRIC FUNCTIONS.

90. To differentiate $\sin^{-1} y$.

Let $\qquad v = \sin^{-1} y, \quad$ then $\quad y = \sin v.$

$\therefore \qquad dy = \cos v \, dv = \sqrt{1 - y^2} dv.$

$\therefore \qquad dv = \frac{dy}{\sqrt{1 - y^2}}, \quad$ or $\quad d(\sin^{-1} y) = \frac{dy}{\sqrt{1 - y^2}}$*.

91. To differentiate $\cos^{-1} y$.

Let $\qquad v = \cos^{-1} y, \quad$ then $\quad y = \cos v.$

$\therefore \qquad dy = - \sin v \, dv = - \sqrt{1 - y^2} dv.$

$\therefore \qquad dv = - \frac{dy}{\sqrt{1 - y^2}}, \quad$ or $\quad d(\cos^{-1} y) = - \frac{dy}{\sqrt{1 - y^2}},$

which always has the sign of $- \sin v$.

---

* To avoid the double sign $\pm$, we shall suppose $0 < v < \frac{\pi}{2}$; for any other quadrant the sign will be that of $\cos v$.

92. To differentiate $\tan^{-1} y$.

Let $\qquad v = \tan^{-1} y, \quad \text{then} \quad y = \tan v.$

$\therefore \qquad dy = \sec^2 v\, dv = (1 + y^2)dv.$

$\therefore \qquad dv = \dfrac{dy}{1+y^2}, \quad \text{or} \quad d(\tan^{-1} y) = \dfrac{dy}{1+y^2}.$

In a similar manner we find

93. $$d(\cot^{-1} y) = -\frac{dy}{1+y^2}.$$

94. $$d(\sec^{-1} y) = \frac{dy}{y\sqrt{y^2-1}}.$$

95. $$d(\operatorname{cosec}^{-1} y) = -\frac{dy}{y\sqrt{y^2-1}}.$$

96. $$d(\operatorname{vers}^{-1} y) = \frac{dy}{\sqrt{2y-y^2}}.$$

EXAMPLES.

Differentiate the following :

1. $y = \sin^{-1}\sqrt{x}.$

$$dy = \frac{d\sqrt{x}}{\sqrt{1-(\sqrt{x})^2}} = \frac{\dfrac{dx}{2\sqrt{x}}}{\sqrt{1-x}} = \frac{dx}{2\sqrt{x-x^2}}.$$

2. $y = \tan^{-1}\left(\dfrac{1-x}{1+x}\right).$

$$dy = \frac{d\left(\dfrac{1-x}{1+x}\right)}{1+\left(\dfrac{1-x}{1+x}\right)^2} = \frac{-\dfrac{2dx}{(1+x)^2}}{\dfrac{(1+x)^2+(1-x)^2}{(1+x)^2}} = -\frac{dx}{1+x^2}.$$

3. $y = \sec^{-1} nx. \qquad dy = \dfrac{dx}{x\sqrt{n^2x^2-1}}.$

4. $y = \operatorname{ver}^{-1}\dfrac{2x}{9}. \qquad dy = \dfrac{dx}{\sqrt{9x-x^2}}.$

5. $y = \sin^{-1}(3x - 4x^3)$. $\quad dy = \dfrac{3dx}{\sqrt{1 - x^2}}$.

6. $y = \sin^{-1}(2x - 1)$. $\quad dy = \dfrac{dx}{\sqrt{x - x^2}}$.

7. $y = \sin^{-1}(\sin x)$. $\quad dy = dx$.

8. $y = \sin^{-1}(\sqrt{\sin x})$. $\quad dy = \frac{1}{2}(\sqrt{1 + \operatorname{cosec} x})dx$.

9. $y = \tan^{-1}\dfrac{2x}{1 - x^2}$. $\quad dy = \dfrac{2dx}{1 + x^2}$.

10. $y = \tan^{-1}\left(\sqrt{\dfrac{1 - \cos x}{1 + \cos x}}\right)$. $\quad dy = \frac{1}{2}dx$.

11. $y = (x + 1)\tan^{-1}\sqrt{x} - \sqrt{x}$. $\quad dy = \tan^{-1}\sqrt{x}\,dx$.

12. $y = \frac{1}{4}\log\dfrac{1 + x}{1 - x} + \frac{1}{2}\tan^{-1}x$. $\quad dy = \dfrac{dx}{1 - x^4}$.

Prove the following by differentiating both sides:

13. $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C.$

14. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C.$

15. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C.$

16. $\displaystyle\int \frac{dx}{\sqrt{2ax - x^2}} = \operatorname{vers}^{-1}\frac{x}{a} + C.$

17. $\displaystyle\int \sqrt{a^2 - x^2}\,dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$

18. $\displaystyle\int \frac{x^2dx}{\sqrt{2ax - x^2}} = -\frac{x + 3a}{2}\sqrt{2ax - x^2} + \frac{3}{2}a^2\operatorname{vers}^{-1}\frac{x}{a} + C.$

19. $\displaystyle\int \frac{madx}{\sqrt{c^2 - b^2 - 2abx - a^2x^2}} = m\sin^{-1}\left(\frac{ax + b}{c}\right) + C.$

20. $\displaystyle\int \frac{madx}{\sqrt{c^2 + b^2 + 2abx + a^2x^2}} = m\log(ax + b + \sqrt{c^2 + (b + ax)^2} + C.$

21. $\displaystyle\int \frac{macdx}{a^2x^2 + 2abx + b^2 + c^2} = m\tan^{-1}\left(\frac{ax + b}{c}\right) + C.$

22. $\int \frac{m(ad - bc)dx}{(ax + b)(cx + d)} = m \log\left(\frac{ax + b}{cx + d}\right) + C.$

The last four equations may be conveniently employed in integrating a certain important class of differentials, of which the following are illustrations:

23. Required the integral of $\frac{5dx}{x^2 - 3x - 28}$.

Here $x^2 - 3x - 28 = (x - 7)(x + 4)$; hence we may integrate by formula 22, thus: Make $ax + b = x - 7$, $cx + d = x + 4$, and $m(ad - bc) = 5$; we then have $a = 1$, $b = -7$, $c = 1$, $d = 4$ and $m = \frac{5}{11}$; hence, substituting in 22, we have

$$\int \frac{5dx}{x^2 - 3x - 28} = \frac{5}{11} \log\left(\frac{x - 7}{x + 4}\right) + C.$$

24. Required the integral of $\frac{3dx}{4x^2 + 3x + 1}$.

Integrating by Ex. 21, we have $a^2 = 4$, $2ab = 3$, $b^2 + c^2 = 1$ and $mac = 3$; whence $a = 2$, $b = \frac{3}{4}$, $c = \frac{1}{4}\sqrt{7}$ and $m = \frac{6}{\sqrt{7}}$.

$$\therefore \int \frac{3dx}{4x^2 + 3x + 1} = \frac{6}{\sqrt{7}} \tan^{-1}\left(\frac{2x + \frac{3}{4}}{\frac{1}{4}\sqrt{7}}\right) + C.$$

25. Find the integral of $\frac{5dx}{\sqrt{16 - 12x - 4x^2}}$.

Integrating by Ex. 19, we have $a^2 = 4$, $2ab = 12$, $c^2 - b^2 = 16$ and $ma = 5$; hence $a = 2$, $b = 3$, $c = 5$, $m = \frac{5}{2}$.

$$\therefore \int \frac{5dx}{\sqrt{16 - 12x - 4x^2}} = \frac{5}{2} \sin^{-1}\left(\frac{2x + 3}{5}\right) + C.$$

26. $\int \frac{dx}{x^2 + x + 1}$. $\qquad \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x + 1}{\sqrt{3}} + C.$

27. $\int \frac{dx}{2 + 5x^2}$. $\qquad \frac{1}{\sqrt{10}} \tan^{-1} \frac{x\sqrt{5}}{\sqrt{2}} + C.$

28. $\displaystyle\int \frac{dx}{2x^2 - 4x - 7}$. $\qquad \dfrac{\sqrt{2}}{12} \log \dfrac{2x - 2 - 3\sqrt{2}}{2x - 2 + 3\sqrt{2}} + C.$

29. $\displaystyle\int \frac{dx}{\sqrt{1 - 3x - x^2}}$. $\qquad \sin^{-1} \dfrac{3 + 2x}{\sqrt{13}} + C.$

30. $\displaystyle\int \frac{dx}{\sqrt{m + nx + rx^2}}$.

$$\frac{1}{\sqrt{r}} \log \left(x\sqrt{r} + \frac{n}{2\sqrt{r}} + \sqrt{m + nx + rx^2}\right) + C.$$

31. $\displaystyle\int \frac{dx}{\sqrt{m + nx - rx^2}}$. $\qquad \dfrac{1}{\sqrt{r}} \sin^{-1} \left(\dfrac{2rx - n}{\sqrt{4mr + n^2}}\right) + C.$

32. $\displaystyle\int \frac{dx}{m + nx + rx^2}$. $\qquad \dfrac{2}{\sqrt{4mr - n^2}} \tan^{-1} \left(\dfrac{2rx + n}{\sqrt{4mr - n^2}}\right) + C.$

97. To find the differential of an arc in polar co-ordinates.

Let AP $(= s)$ be the arc of a curve, O the pole, OP $(= r)$ the radius vector, and PT a tangent to the curve at P.

Let $\theta = XOP$ and $\psi = OPT$. Increase θ by POP' $(= \Delta\theta)$, then arc $PP' = \Delta s$ and $OP' = r + \Delta r$. Draw PD perpendicular to OP', then $PD = r \sin \Delta\theta$ and $DP' = r + \Delta r - r \cos \Delta\theta$.

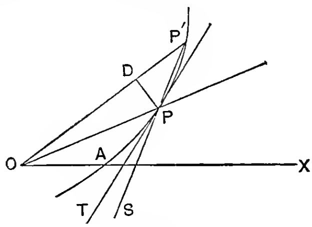

FIG. 17.

The chord $PP' = \sqrt{PD^2 + DP'^2}$,

$$\therefore \qquad \frac{\text{chord } PP'}{\Delta\theta} = \sqrt{\left(\frac{PD}{\Delta\theta}\right)^2 + \left(\frac{DP'}{\Delta\theta}\right)^2}.$$

Passing to the limit, remembering that as $\Delta\theta$ approaches 0, the limits of $\dfrac{\text{chord } PP'}{\text{arc } PP'}$ and $\dfrac{\sin \Delta\theta}{\Delta\theta}$ (Art. 33), also of $\cos \Delta\theta$, is each unity, we have

$$ds = \left(\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\right) d\theta.$$

98. COR. I. As P' approaches P, the angle $OP'P$ approaches the angle OPT $(=\psi)$; therefore

$$\tan\psi = \underset{\Delta\theta \doteq 0}{\text{limit}}\left[\frac{PD}{P'D}\right] = \frac{rd\theta}{dr}, \quad \sin\psi = \frac{rd\theta}{ds}, \quad \cos\psi = \frac{dr}{ds}.$$

FUNCTIONS OF TWO OR MORE INDEPENDENT VARIABLES.

99. A function of two variables, as $u = x^2y + y^3$ and $u = \sin(x+y)$, is represented by $f(x, y)$; and similarly $f(x, y, z)$ represents a function of the three variables x, y and z.

Since x and y are independent of each other, the function $u = f(x, y)$ may change in three ways. Thus, let $u = xy =$ area of $OBPA$, where $OB = x$ and $OA = y$. (1) x may change and y not, which would give $du = BCDP = ydx$; (2) y may change and x not, which would give $du = APFG = xdy$; (3) both x and y may change, which would give (understanding by du the portion of the increment which is of the first degree in dx and dy) $du = ydx + xdy$. Hence du may have three different values, and it is desirable to employ a notation by which they may be represented.

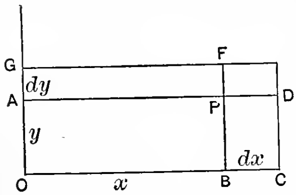

FIG. 18.

100. A **Partial Differential** of a function of two or more variables is the differential obtained on the hypothesis that only one of its variables changes, as ydx and xdy in the previous example, and these are denoted respectively by $\frac{du}{dx}dx$ and $\frac{du}{dy}dy$, or d_xu and d_yu.

101. Total Differential. In a function of two or more independent variables, if each variable receives an increment, that portion of the corresponding increment of the function which is of the first degree with respect to the increments of the variables is the total differential of the function.

102. Prop. *The total differential of a function of two or more independent variables is the sum of its partial differentials.*

Let u represent any function of x and y.

When x becomes $x+h$, the part of the corresponding increment of u which involves the first power of h or dx is $\frac{du}{dx}dx$. Hence, omitting the terms involving the higher powers of dx, Art. 29, the new value of u is

$$u+\frac{du}{dx}dx.$$

In this new value of u when y is increased by k $(=dy)$ the parts of the corresponding increments of u and $\frac{du}{dx}dx$ which involve only the first power of dy are $\frac{du}{dy}dy$ and $\frac{d}{dy}\left(\frac{du}{dx}dx\right)dy$; hence, omitting the terms involving the higher powers of dy, the second new value of u is

$$u+\frac{du}{dx}dx+\frac{du}{dy}dy+\frac{d}{dy}\left(\frac{du}{dx}dx\right)dy,$$

which result is the same as if x and y had changed simultaneously, for the result of increasing x by dx and y by dy is evidently the same whether the changes be made separately or simultaneously. Hence, since the last term involves the product of dx and dy, we have

$$du=\frac{du}{dx}dx+\frac{du}{dy}dy.$$

In a similar manner it may be shown that the theorem is true of functions having any number of variables.

Cor. I. The total differential of a function is the sum of those parts of its increment which vary as the increments of the variables, respectively.

Cor. II. The theorem is also true of functions whose variables are not independent of each other.

COR. III. The total differential of a function of two or more variables, as x, y and z, is the sum of the differentials obtained by first differentiating as though x only were variable, and then as though y only were variable, and then as though z only were variable.

103. A **Partial Derivative** of a function of two or more variables is the ratio of the partial differential of the function to the differential of the variable supposed to change.

104. The **Total Derivative** of a function of two or more variables, only one of which is independent, is the ratio of the total differential of the function to the differential of the independent variable.

To prevent confusion we shall distinguish the total differential or derivative of two or more variables by enclosing it in brackets.

Thus, if $u = f(x, y)$, $\quad [du] = \frac{du}{dx}dx + \frac{du}{dy}dy,$

where $\frac{du}{dx}dx$ and $\frac{du}{dy}dy$ are the partial differentials with respect to x and y, respectively.

EXAMPLES.

Find the total differential of—

1. $u = x^2 - 3xy + 2y^2$.

$$\frac{du}{dx}dx = (2x - 3y)dx; \quad \frac{du}{dy}dy = -(3x - 4y)dy.$$

$$\therefore [du] = (2x - 3y)dx - (3x - 4y)dy.$$

2. $u = \frac{x+y}{x-y}$. $\qquad [du] = \frac{2(xdy - ydx)}{(x-y)^2}$.

3. $u = \sin^{-1}\frac{x}{y}$. $\qquad [du] = \frac{ydx - xdy}{y\sqrt{y^2 - x^2}}$.

4. $u = \tan^{-1}\frac{y}{x}$. $\qquad [du] = \frac{xdy - ydx}{x^2 + y^2}$.

5. $u = \log y^x$. $\qquad [du] = \frac{x}{y} dy + \log y dx.$

6. $u = y^{\sin x}$. $\qquad [du] = y^{\sin x} \log y \cos x\, dx + \frac{\sin x}{y^{\text{covers}\, x}} dy.$

105. Function of Functions. If $u = F(y)$ and $y = f(x)$, u is indirectly a function of x through y. In such cases the value of $\frac{du}{dx}$ may be obtained by finding the value of u in terms of x, and differentiating the result; but it is often more easily found by the formula

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx}. \quad \ldots \ldots \ldots \quad (1)$$

That is, the derivative of u with respect to x is equal to the derivative of u with respect to y multiplied by the derivative of y with respect to x.

Thus, if $u = \tan^{-1} y$, and $y = \log x$, then

$$\frac{du}{dy} = \frac{1}{1+y^2}, \quad \frac{dy}{dx} = \frac{1}{x}, \quad \text{and} \quad \therefore \frac{du}{dx} = \frac{1}{x(1+y^2)}.$$

If $u = F(v, y)$, $v = f(x)$ and $y = f_1(x)$, to obtain $\left[\frac{du}{dx}\right]$, the total derivative of u with respect to x, we may proceed thus:

Since $$[du] = \frac{du}{dv}dv + \frac{du}{dy}dy,$$

dividing by dx, $$\left[\frac{du}{dx}\right] = \frac{du}{dv}\frac{dv}{dx} + \frac{du}{dy}\frac{dy}{dx}, \quad \ldots \quad (2)$$

which gives $\frac{du}{dx}$ in terms of derivatives which can be reckoned out from the given equations.

Thus, if $u = v^2 + vy$, $v = \log x$ and $y = e^x$, then

$$\frac{du}{dv} = 2v + y, \quad \frac{du}{dy} = v, \quad \frac{dv}{dx} = \frac{1}{x} \quad \text{and} \quad \frac{dy}{dx} = e^x;$$

substituting in (2), we have

$$\frac{du}{dx} = \frac{2v + y}{x} + ve^x.$$

If $u = F(x, v, z)$, $v = f(x)$ and $z = f_1(x)$, we have

$$[du] = \frac{du}{dx}dx + \frac{du}{dv}dv + \frac{du}{dz}dz.$$

$$\therefore \qquad \left[\frac{du}{dx}\right] = \frac{du}{dx} + \frac{du}{dv}\frac{dv}{dx} + \frac{du}{dz}\frac{dz}{dx}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $\frac{du}{dx}$, $\frac{du}{dv}$ and $\frac{du}{dz}$ are partial derivatives, and $\left[\frac{du}{dx}\right]$ the total derivative of u.

EXAMPLES.

Find $\left[\frac{du}{dx}\right]$ in the following:

1. $u = e^x(y - z)$, $y = \sin x$, and $z = \cos x$.

$$\left[\frac{du}{dx}\right] = 2e^x \sin x.$$

2. $u = \tan^{-1}\frac{y}{x}$, and $y = \sqrt{r^2 - x^2}$. $\qquad \left[\frac{du}{dx}\right] = -\frac{1}{\sqrt{r^2 - x^2}}$.

3. $u = \tan^{-1}(xy)$, and $y = e^x$. $\qquad \left[\frac{du}{dx}\right] = \frac{e^x(1 + x)}{1 + x^2 e^{2x}}$.

4. If $y = uz$ and $u = e^x$, $z = x^4 - 4x^3 + 12x^2 - 24x + 24$, find the slope of the curve of which y is the ordinate and x the abscissa.

$$\frac{dy}{dx} = e^x x^4.$$

106. Successive Partial Differentials and Derivatives. We have seen that the differential of u $(= f(x, y))$ (1) with respect to x is denoted by $\frac{d}{dx}(u)dx$, and (2) with respect to y by $\frac{d}{dy}(u)dy$.

Similarly, the differential of $\frac{du}{dx}dx$

(1) with respect to x is denoted by $\frac{d}{dx}\left(\frac{du}{dx}dx\right)dx, = \frac{d^2u\,dx^2}{dx^2}$;

(2) with respect to y is denoted by $\frac{d}{dy}\left(\frac{du}{dx}dx\right)dy, = \frac{d^2u\,dx\,dy}{dx\,dy}$.

Hence $\frac{d^2u\,dy\,dx}{dy\,dx}$ is a symbol for the result obtained by differentiating u two times in succession: once, and first, with respect to y, and then once with respect to x. Similarly, $\frac{d^3u}{dx\,dy^2}dx\,dy^2$ indicates the result of three successive differentials of u: first, once with respect to x, and then twice with respect to y.

In finding these successive partial differentials, we treat dy and dx as constants, since y and x are regarded as independent variables, see Art. 68.

The symbols for the partial derivatives are

$$\frac{d^2u}{dx^2},\quad \frac{d^2u}{dx\,dy},\quad \frac{d^2u}{dy^2},\quad \frac{d^3u}{dx^3},\quad \frac{d^3u}{dx^2\,dy},\ \text{etc.}$$

107. Principle. If $u = f(x, y)$, $\frac{d^2u}{dx\,dy} = \frac{d^2u}{dy\,dx}$.

For, Art. 102, changing x and then y to obtain $\frac{d^2u}{dx\,dy}$, or changing y and then x to obtain $\frac{d^2u}{dy\,dx}$, is equivalent to changing x and y simultaneously; and therefore the results are equal.

Cor. I. If u be differentiated m times with respect to x, and n times with respect to y, the result is the same whatever be the order of the differentiations.

EXAMPLES.

1. Given $u = x^2y^3$; find $\frac{d^2u}{dx\,dy}$ and $\frac{d^2u}{dy\,dx}$. $6xy^2$.

2. Given $u = x^2y + xy^3$; verify $\frac{d^3u}{dx\,dy^2} = \frac{d^3u}{dy^2\,dx}$.

3. If $u = y \log(1 + xy)$, show that $\frac{d^2u}{dy\,dx} = \frac{d^2u}{dx\,dy}$.

4. Given $u = \frac{x^2 - y^2}{x^2 + y^2}$; verify $\frac{d^2u}{dx\,dy} = \frac{d^2u}{dy\,dx}$.

108. To find the successive differentials of a function of two independent variables.

Let $u = f(x, y)$; then

$$[du] = \frac{du}{dx}dx + \frac{du}{dy}dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Differentiating (1) and observing that $\frac{du}{dx}$ and $\frac{du}{dy}$ are usually functions of x and y, and that x and y are independent, Art. 68, we get

$$[d^2u] = \frac{d^2u}{dx^2}dx^2 + \frac{d^2u}{dx\,dy}dx\,dy + \frac{d^2u}{dy\,dx}dy\,dx + \frac{d^2u}{dy^2}dy^2,$$

or, Art. 107,

$$[d^2u] = \frac{d^2u}{dx^2}dx^2 + 2\frac{d^2u}{dx\,dy}dx\,dy + \frac{d^2u}{dy^2}dy^2. \quad . \quad . \quad . \quad . \quad (2)$$

Differentiating (2), remembering that each term is a function of x and y, we have

$$[d^3u] = \frac{d^3u}{dx^3}dx^3 + 3\frac{d^3u}{dx^2\,dy}dx^2\,dy + 3\frac{d^3u}{dx\,dy^2}dx\,dy^2 + \frac{d^3u}{dy^3}dy^3 \ldots;$$

and similarly may $[d^4u]$, $[d^5u]$, etc., be found. By observing the analogy between the values of $[d^2u]$ and $[d^3u]$, and the developments of $(a + x)^2$ and $(a + x)^3$, the formula for the value of $[d^nu]$ may be easily written out.

109. Implicit Functions. In functions of the form $f(x, y) = 0$, the formula $[du] = \frac{du}{dx}dx + \frac{du}{dy}dy$ is often useful in finding the value of the derivative, or slope, $\frac{dy}{dx}$.

Thus, take $\quad f(x, y) = ax^3 + x \sin y = 0.$

Making $u = ax^3 + x \sin y$, we have

$$[du] = \frac{du}{dx}dx + \frac{du}{dy}dy = (3ax^2 + \sin y)dx + x \cos y \, dy.$$

But since $u = 0$, $[du] = 0$.

$$\frac{dy}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dy}} = -\frac{3ax^2 + \sin y}{x \cos y}.$$

EXAMPLES.

Find the derivative, $\frac{dy}{dx}$, of the following:

1. $(y - b)^2 - x^3 + ax^2 = 0.$ $\qquad \frac{dy}{dx} = \frac{3x^2 - 2ax}{2(y - b)}.$

2. $x^4 + 2ax^2y - ay^3 = 0.$ $\qquad \frac{dy}{dx} = \frac{4x^3 + 4axy}{3ay^2 - 2ax^2}.$

3. $x^3 + 3axy + y^3 = 0.$ $\qquad \frac{dy}{dx} = -\frac{x^2 + ay}{y^2 + ax}.$

110. Successive Derivatives of an Implicit Function. The following examples will serve to illustrate how the successive derivatives of implicit functions in general may be determined.

EXAMPLES.

1. Find $\frac{d^2y}{dx^2}$ when $y^2 - 4ax = 0.$

Here $$\frac{dy}{dx} = \frac{2a}{y}. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

Differentiating (1), we have

$$\frac{d^2y}{dx} = \frac{-2ady}{y^2}. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

Eliminating dy in (1) and (2), we have

$$\frac{d^2y}{dx^2} = -\frac{4a^2}{y^3}.$$

2. Given $a^2y^2 + b^2x^2 - a^2b^2 = 0$, show that $\dfrac{d^2y}{dx^2} = -\dfrac{b^4}{a^2y^3}$.

3. Given $y^3 + x^3 - 3axy = 0$, show that $\dfrac{d^2y}{dx^2} = -\dfrac{2a^3xy}{(y^2 - ax)^3}$.

111. Change of the Independent Variable. After obtaining the derivatives $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}$, etc., on the hypothesis that x was the independent variable and y the function, it is sometimes desirable to change these expressions into their equivalents with y for the independent variable and x the function, or with x and y for the functions and some other variable, as t, for the independent variable, and so on.

112. To find the successive derivatives of $\dfrac{dy}{dx}$ when neither x nor y is independent.

Under this hypothesis $\dfrac{dy}{dx}$ is to be differentiated as a fraction having both terms variable.

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dx\,d^2y - dy\,d^2x}{dx^3}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Similarly,
$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

$$= \frac{(dx\,d^3y - dy\,d^3x)dx - 3(dx\,d^2y - dy\,d^2x)d^2x}{dx^5}. \quad . \quad (2)$$

In like manner we obtain the other successive derivatives.

Cor. I. If y is independent, then $d^2y = d^3y = 0$, and we have

$$\frac{d^2y}{dx^2} = -\frac{dy\,d^2x}{dx^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{d^3y}{dx^3} = \frac{3(d^2x)^2dy - d^3x\,dy\,dx}{dx^5}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Formulas (1) and (2) give us the values to be substituted for $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ when neither x nor y is independent; and formulas (3) and (4) give us the values of the same derivatives when y is independent.

If a new variable t, of which $x = f(t)$, is to be the independent variable, in Art. 111, we replace x, dx, d^2x, etc., by their values as determined from $x = f(t)$.

EXAMPLES.

1. Given $yd^2y + dy^2 + dx^2 = 0$, where x is independent, to find (1) the transformed equation in which neither x nor y is independent; also (2) the one in which y is independent.

(1) Dividing by dx^2, substituting for $\frac{d^2y}{dx^2}$ from (1), and multiplying both members by dx^3, we have

$$y(d^2y\,dx - d^2x\,dy) + dy^2\,dx + dx^3 = 0.$$

(2) Making $d^2y = 0$ in this last equation, and dividing by $-dy^3$, we have

$$y\frac{d^2x}{dy^2} - \frac{dx^3}{dy^3} - \frac{dx}{dy} = 0.$$

2. Change the independent variable from x to t in $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$, when $x = 2\sqrt{t}$.

Substituting for $\frac{d^2y}{dx^2}$ from (1), multiplying by $x\,dx^3$, and making $x = 2\sqrt{t}$, $dx = t^{-\frac{1}{2}}dt$, and $d^2x = -\frac{1}{2}t^{-\frac{3}{2}}dt^2$, we obtain

$$t\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0.$$

Change the independent variable from x to y in the two following equations:

3. $3\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx}\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2 = 0.$ $\qquad \frac{d^3x}{dy^3} + \frac{d^2x}{dy^2} = 0.$

4. $x\frac{d^2y}{dx^2} + \frac{dy^3}{dx^3} - \frac{dy}{dx} = 0.$ $\qquad x\frac{d^2x}{dy^2} + \frac{dx^2}{dy^2} - 1 = 0.$

Change the independent variable from x to t in the two following equations:

5. $\dfrac{d^2y}{dx^2} + \dfrac{2x}{1+x^2}\dfrac{dy}{dx} + \dfrac{y}{(1+x^2)^2} = 0$, where $x = \tan t$.

$$\frac{d^2y}{dt^2} + y = 0.$$

6. $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 0$, where $x = \cos t$. $\qquad \dfrac{d^2y}{dt^2} = 0.$

7. Find the value of $R = \left(1 + \dfrac{dy^2}{dx^2}\right)^{\frac{3}{2}} \div \dfrac{d^2y}{dx^2}$, where x is independent, supposing neither x nor y to be independent.

$$R = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx\, d^2y - dy\, d^2x}.$$

MISCELLANEOUS EXAMPLES.

Differentiate the following:

1. $y = 3(x^2+1)^{\frac{4}{3}}(4x^2-3)$. $\qquad dy = 56x^3(x^2+1)^{\frac{1}{3}}dx.$

2. $y = \dfrac{1}{x + \sqrt{1+x^2}}$. $\qquad \dfrac{dy}{dx} = \dfrac{x}{\sqrt{1+x^2}} - 1.$

3. $y = \dfrac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}}$. $\qquad \dfrac{dy}{dx} = -\dfrac{a^2 + a\sqrt{a^2-x^2}}{x^2\sqrt{a^2-x^2}}.$

4. $y = x + \log\cos\left(\dfrac{\pi}{4} - x\right)$. $\qquad \dfrac{dy}{dx} = \dfrac{2}{1+\tan x}.$

5. $y = \log\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$. $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{a}}{(a-x)\sqrt{x}}.$

6. $y = \dfrac{x\log x}{1-x} + \log(1-x)$. $\qquad \dfrac{dy}{dx} = \dfrac{\log x}{(1-x)^2}.$

7. $y = \log\sqrt{\dfrac{x^2-x+1}{x^2+x+1}}$. $\qquad \dfrac{dy}{dx} = \dfrac{x^2-1}{x^4+x^2+1}.$

8. $y = \sin(x+a)\cos(x-a)$. $\qquad dy = \cos 2x dx.$

9. $y = \log\tan\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right)$. $\qquad \dfrac{dy}{dx} = \sec x.$

10. $\sin 2x = 2\sin x\cos x$. $\qquad \cos 2x = \cos^2 x - \sin^2 x.$

11. $\sin 2x = \dfrac{2 \tan x}{1 + \tan^2 x}$. $\quad \cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$.

12. $\sin 3x = 3 \sin x - 4 \sin^3 x$. $\quad \cos 3x = 4 \cos^3 x - 3 \cos x$.

13. $y = \tan^{-1} e^x$. $\quad \dfrac{dy}{dx} = \dfrac{1}{e^x + e^{-x}}$.

14. $y = \cos^{-1}\left(\dfrac{x^6 - 1}{x^6 + 1}\right)$. $\quad \dfrac{dy}{dx} = -\dfrac{6x^2}{x^6 + 1}$.

15. $y = \sec^{-1} \dfrac{1}{2x^2 - 1}$. $\quad \dfrac{dy}{dx} = -\dfrac{2}{\sqrt{1 - x^2}}$.

16. $y = \tan^{-1} x + \tan^{-1} \dfrac{1 - x}{1 + x}$. $\quad \dfrac{dy}{dx} = 0$.

Prove the following by differentiation:

17. $\displaystyle\int \frac{n dx}{\cos^2 x + n^2 \sin^2 x} = \tan^{-1} (n \tan x) + C.$

18. $\displaystyle\int \frac{2ax^2 dx}{x^4 - a^4} = \tan^{-1} \frac{x}{a} + \frac{1}{2} \log \left(\frac{x - a}{x + a}\right) + C.$

19. $\log (1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \text{etc.}$

20. $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \text{etc.}$

Find the slopes of the following curves:

21. The quadratrix, $y = (a - x) \tan \dfrac{\pi x}{2a}$.

$$\frac{dy}{dx} = \frac{\pi}{2a} (a - x) \sec^2 \frac{\pi x}{2a} - \tan \frac{\pi x}{2a}.$$

22. The cycloid, $x = r \operatorname{vers}^{-1} \dfrac{y}{r} - \sqrt{2ry - y^2}$.

$$\frac{dy}{dx} = \sqrt{\frac{2r - y}{y}}.$$

23. The catenary, $y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)$. $\qquad \frac{dy}{dx} = \frac{1}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right)$.

24. The tractrix, $x = a \log \frac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}$.

$$\frac{dy}{dx} = -\frac{y}{\sqrt{a^2 - y^2}}.$$

Find the following:

25. $\int\left(\frac{12}{x^3} - \frac{5}{x^4}\right)dx$. $\qquad -\frac{6}{x^2} + \frac{5}{3x^3} + C$.

26. $\int\frac{4x\,dx}{3(x^2+2)^{\frac{1}{3}}}$. $\qquad (x^2+2)^{\frac{2}{3}} + C$.

27. $\int\left(\frac{8}{3}x^{\frac{5}{3}} - 5x^{\frac{2}{3}} + 2x^{-\frac{1}{3}} + \frac{1}{3}x^{-\frac{4}{3}}\right)dx$. $\qquad \frac{(x-1)^3}{\sqrt[3]{x}} + C$.

28. $\int\frac{\sqrt{a^2-x^2}}{x^4}dx$. $\qquad -\frac{(a^2-x^2)^{\frac{3}{2}}}{3a^2x^3} + C$.

29. $\int\frac{dx}{(a^2-x^2)^{\frac{3}{2}}}$. $\qquad \frac{x}{a^2\sqrt{a^2-x^2}} + C$.

30. $\int\frac{dx}{x\sqrt{2ax-x^2}}$. $\qquad -\frac{\sqrt{2ax-x^2}}{ax} + C$.

31. $\int\frac{(x^2-2)^3dx}{x^5}$. $\qquad \frac{2}{x^4} - \frac{6}{x^2} + \frac{x^2}{2} - 6\log x + C$.

32. $\int\frac{2(x+1)dx}{x^2+2x+3}$. $\qquad \log(x^2+2x+3) + C$.

33. $\int\frac{(2x-1)}{2x+3}dx$. $\qquad x - 2\log(2x+3) + C$.

34. $\int\frac{dx}{9x^2-4}$. $\qquad \frac{1}{12}\log\left(\frac{3x-2}{3x+2}\right) + C$.

35. $\int\frac{dx}{\sqrt{1+4x^2}}$. $\qquad \frac{1}{2}\log\left(2x+\sqrt{1+4x^2}\right) + C$.

36. $\int\frac{dx}{\sqrt{x^2-4x+13}}$. $\qquad \log\left(x-2+\sqrt{x^2-4x+13}\right) + C$.

37. $\int\frac{2dx}{3x^2+10x+3}$. $\qquad \frac{1}{4}\log\left(\frac{3x+1}{x+3}\right) + C$.

38. $\int \frac{dx}{\sqrt{1+3x-x^2}}$. $\sin^{-1}\frac{2x-3}{\sqrt{13}}+C$.

39. $\int \frac{dx}{x^2-6x+11}$. $\frac{1}{\sqrt{2}}\tan^{-1}\frac{x-3}{\sqrt{2}}+C$.

40. $\int_0^1 (x^2-2x+2)(x-1)dx$. $-\frac{3}{4}$.

41. $\int_2^3 \frac{3xdx}{2\sqrt[4]{x^2-4}}$. $\sqrt[4]{125}$.

42. $\int_0^2 \frac{x^3dx}{x+1}$. $2\frac{2}{3}-\log 3$.

43. $\int 3\sin^2 x\cos xdx$. $\sin^3 x + C$.

44. $\int 3b(a-b\cos^2 x)^{\frac{1}{2}}\sin x\cos xdx$. $(a-b\cos^2 x)^{\frac{3}{2}}+C$.

45. $\int \frac{\sec^2 x+1}{\tan x+x}dx$. $\log(\tan x+x)+C$.

46. $\int (\tan x+\cot x)^2dx$. $\tan x-\cot x+C$.

---

47. Find $\frac{d^2y}{dx^2}$ when $y^2-2xy+c=0$. $y\frac{(y-2x)}{(y-x)^3}$.

48. Given $y^2-2axy+x^2=c$, to find $\frac{d^2y}{dx^2}$. $\frac{c(a^2-1)}{(y-ax)^3}$.

49. In $yd^2y+dy^2+dx^2=0$, change the independent variable from x to y.

$$y\frac{d^2x}{dy^2}-\frac{dx^3}{dy^3}-\frac{dx}{dy}=0.$$

50. Change the independent variable from x to z in $(2x-1)^3\frac{d^3y}{dx^3}+(2x-1)\frac{dy}{dx}=2y$, where $2x=1+e^z$.

$$4\frac{d^3y}{dz^3}-12\frac{d^2y}{dz^2}+9\frac{dy}{dz}=y.$$

51. If $y^2=\sec 2x$, prove that $\frac{d^2y}{dx^2}=3y^5-y$.

52. Given $s = t^3 - 5t^2 + 6t$; find the velocity (v) and its rate of change (a') when $t = 10$. $v = 206$; $a' = 50$.

53. In t seconds after a body leaves a certain point the rate of change of its velocity is $6t - 12$ feet per second; required its velocity and distance from the point of starting.

$v = 3t^2 - 12t$; $s = t^3 - 6t^2$.

54. In the last example how far will the body travel in 10 seconds?

$-\left[s\right]_0^4 + \left[s\right]_4^{10} = 32 + 432 = 464$ (ft.).

55. How many times faster is x increasing than $\log x$, when $x = n$? n times.

56. Required the value of x at the point where the slope of the curve $y = \tan x$ is 2. $\frac{\pi}{4}$.

57. A man is walking on a straight path at the rate of 5 ft. per second; how fast is he approaching a point 120 ft. from the path in a perpendicular, when he is 50 ft. from the foot of the perpendicular? $1\frac{12}{13}$ ft. per sec.

58. A vertical wheel whose circumference is 20 ft. makes 5 revolutions a second about a fixed axis. How fast is a point in its circumference moving horizontally, when it is 30° from either extremity of the horizontal diameter? 50 ft. per sec.

59. A buggy wheel whose radius is r rolls along a horizontal path with a velocity v'; required the velocity $\left(\frac{ds}{dt}\right)$ of any point (x, y) in its circumference; also the velocity of the point horizontally $\left(\frac{dx}{dt}\right)$ and vertically $\left(\frac{dy}{dt}\right)$.

The curve described by the point in the circumference of the wheel is a cycloid whose equation is $x = r \text{ vers}^{-1} \frac{y}{r} - \sqrt{2ry - y^2}$; differentiating this and dividing by dt, we have

$$\frac{dx}{dt} = \frac{y}{\sqrt{2ry - y^2}} \frac{dy}{dt}. \quad \ldots \ldots \ldots \quad (1)$$

Again, the abscissa of the center of the wheel is $r \text{ vers}^{-1} \dfrac{y}{r}$; differentiating this and dividing by dt, and we have

$$v' = \frac{r}{\sqrt{2ry - y^2}} \frac{dy}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again, since $ds^2 = dx^2 + dy^2$, we have

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (1), (2), and (3) we readily obtain

$$\frac{dy}{dt} = \left(\frac{\sqrt{2ry - y^2}}{r}\right)v', \quad \frac{dx}{dt} = \left(\frac{y}{r}\right)v', \quad \text{and} \quad \frac{ds}{dt} = \left(\sqrt{\frac{2y}{r}}\right)v'.$$

60. In the last example find the values of $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$, and $\dfrac{ds}{dt}$ at the point (1) where $y = 0$; (2) where $y = r$; (3) where $y = 2r$.

61. Water is poured at a uniform rate into a conical glass 3 inches in height, filling the glass in 8 seconds. At what rate is the surface rising (1) at the end of 1 second? (2) At what rate when the surface reaches the brim?

(1) $\frac{1}{2}$ in. per sec. (2) $\frac{1}{8}$ in. per sec.

CHAPTER V.

SERIES, DEVELOPMENT OF FUNCTIONS, AND INDETERMINATE FORMS.

SERIES.

113. A **Series** is a succession of terms following one another according to some fixed law.

If the sum of the first n terms of an infinite series approaches a definite limit as n increases indefinitely, the series is **Convergent**; if not, it is **Divergent**.

The sum of a finite series is the sum of all its terms; and the sum of an infinite convergent series is the limit which the sum of the first n terms approaches as n increases. An infinite divergent series has no definite sum.

114. To **Develop** a function is to find a series, the sum of which shall be equal to the function. Hence the development of a function is either a finite or an infinite convergent series.

For example, $(x+1)^3 = x^3 + 3x^2 + 3x + 1.$

This finite series is the development of the function $(x+1)^3$ for any value of x.

Again, by division we obtain

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots x^{n-1}. \quad . \quad . \quad . \qquad (1)$$

Now this series is the development of $\frac{1}{1-x}$ only for values

of x numerically less than 1, for the omitted remainder is $\frac{x^n}{1-x}$; hence, denoting the series by s, we have

$$\frac{1}{1-x} = s + \frac{x^n}{1-x}.$$

Therefore s can be the value of $\frac{1}{1-x}$ only when $\frac{x^n}{1-x} = 0$, and s the development of $\frac{1}{1-x}$ only for such values of x as will cause $\frac{x^n}{1-x}$ to approach 0, as n increases indefinitely, and this can be the case only when $x < 1$.

Thus: (1) For $x = 2$ we have

$$-1 = 1 + 2 + 4 + 8 + \ldots 2^{n-1} - 2^n,$$

which would be *absurd* were the remainder -2^n omitted.

(2) For $x = \frac{1}{2}$ we have

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \frac{1}{2^{n-1}} + \frac{1}{2^n},$$

in which the remainder $\frac{1}{2^n}$ decreases as n increases, indefinitely.

115. A series is said to be *absolutely* convergent when it remains convergent on making the signs of all its terms positive; but only *conditionally* convergent when it becomes divergent on such a change of signs.

The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is an example of a conditionally convergent series.

116. Prop. *The infinite series* $u_1 + u_2 + \ldots u_{n-1} + u_n + \ldots$ *will be absolutely convergent if the terms are all finite, and the limit of the ratio* $\frac{u_n}{u_{n-1}}$, *as* n *is indefinitely increased, is numerically less than unity.*

Since the limit of $\frac{u_n}{u_{n-1}}$, as n is indefinitely increased, is less than 1, there must exist some finite integer, c, such that for all values of n which are greater than c, $\frac{u_n}{u_{n-1}}$ is less than 1.

The sum of the terms $u_1 + u_2 + \ldots u_c$ is a definite finite quantity, and it only remains to show that the sum of the remaining terms, $u_{c+1} + u_{c+2} + \ldots$, is also.

Let r be a number less than 1 but greater than any of the ratios $\frac{u_{c+1}}{u_c}, \frac{u_{c+2}}{u_{c+1}}, \frac{u_{c+3}}{u_{c+2}}, \ldots$; then

$$u_{c+1} < r\,u_c,$$
$$u_{c+2} < r\,u_{c+1} < r^2 u_c,$$
$$u_{c+3} < r\,u_{c+2} < r^2 u_{c+1} < r^3 u_c,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots;$$

$$\therefore \quad u_{c+1} + u_{c+2} + u_{c+3} + \ldots < u_c(1 + r + r^2 + \ldots)r,$$

or $$u_{c+1} + u_{c+2} + u_{c+3} + \ldots < r\,u_c\left(\frac{1}{1-r}\right),$$

the last member of which is finite, since $r < 1$, Art. 114; therefore the first member is also finite.

Again, that the sum $u_{c+1} + u_{c+2} + \ldots$, though finite, may be indeterminate, is precluded by the fact that the limit of u_n, as n approaches ∞, is 0. Therefore the series is convergent.

SCHOLIUM. The limit of u_n as n increases is always 0 in case of a convergent series, but the mere fact that the limit of its nth term is 0 does not prove a series to be convergent.

117. COR. I. The series $a_0 + a_1x + a_2x^2 + \ldots a_{n-1}x^{n-1} + a_nx^n + \ldots$, where a_0, a_1, a_2, etc., are independent of x, is convergent for all values of x numerically less than k (say), the limit of $\frac{a_{n-1}}{a_n}$, as n approaches ∞.

For, by Art. 116, the series is convergent if the limit of

$a_n x^n \div a_{n-1}x^{n-1}$, or $a_n x \div a_{n-1}$, is less than 1, and therefore if $x < k$.

118. COR. II. When $x > k$ the series is divergent, and when $x = k$ the series in some cases is convergent, and in others divergent.

119. COR. III. The series $f'(x) = a_1 + 2a_2x + 3a_3x^2 + \ldots (n-1)a_{n-1}x^{n-2} + na_nx^{n-1}$, obtained by differentiating $f(x) = a_0 + a_1x + a_2x^2 + \ldots a_{n-1}x^{n-1} + a_nx^n$, is convergent for the same values of x as the last-mentioned series.

For in the former $k =$ the limit of $\dfrac{(n-1)a_{n-1}}{na_n}$, which is the same as the limit of $\dfrac{a_{n-1}}{a_n}$.

It is also evident that the limits of convergence of the series obtained by integrating the individual terms of $f(x)dx$ are the same as those of the series $f(x)$ itself.

EXAMPLES.

Find the values of x which will render the following convergent:

1. $1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \ldots \dfrac{x^{n-1}}{n-1} + \dfrac{x^n}{n} + \ldots$

Here $a_{n-1} = \dfrac{1}{n-1}$ and $a_n = \dfrac{1}{n}$. $\therefore \dfrac{a_{n-1}}{a_n} = \dfrac{n}{n-1} = 1 + \dfrac{1}{n-1}$, which $= 1$ when $n = \infty$. Hence (117), $-1 < x < 1$; that is, x lies between -1 and $+1$.

2. $1 + x + \dfrac{x^2}{\lfloor 2} + \dfrac{x^3}{\lfloor 3} + \dfrac{x^4}{\lfloor 4} \ldots \dfrac{x^{n-1}}{\lfloor n-1} + \dfrac{x^n}{\lfloor n} + \ldots$

Here $\dfrac{a_{n-1}}{a_n} = \dfrac{\lfloor n}{\lfloor n-1} = n$, which $= \infty$ when $n = \infty$; hence $-\infty < x < \infty$; that is, the series is convergent for all finite values of x.

3. $\dfrac{3x}{2} + \dfrac{5x^2}{5} + \dfrac{7x^3}{10} + \ldots + \dfrac{2(n-1)+1}{(n-1)^2+1}x^{n-1} + \dfrac{2n+1}{n^2+1}x^n + \ldots$

Here $\frac{a_{n-1}}{a_n} = \frac{2(n-1)+1}{(n-1)^2+1} \times \frac{n^2+1}{2n+1}$, which $= 1$ when $n = \infty$.

$\therefore\ -1 < x < 1.$

4. $x + 2^2x^2 + 3^2x^3 + \ldots (n-1)^2x^{n-1} + n^2x^n + \ldots$

$-1 < x < 1.$

5. $\frac{x}{1+\sqrt{1}} + \frac{x^2}{1+\sqrt{2}} + \ldots \frac{x^{n-1}}{1+\sqrt{n-1}} + \frac{x^n}{1+\sqrt{n}} + \ldots$

$-1 < x < 1.$

DEVELOPMENT OF FUNCTIONS.

120. There are two common and useful formulas for developing functions: Taylor's and Maclaurin's.

We shall deduce Taylor's formula first, as Maclaurin's may be derived from it.

121. Taylor's Formula is a formula for developing $f(y+x)$ in a series, where $f(y+x)$ represents any function of the *sum* of two variables, such as $(y+x)^n$, $\log(y+x)$, $\sin(y+x)$, a^{y+x}.

The derivation of Taylor's formula may be regarded simply as the process of finding the acceleration of a function. Thus, let $u = f(y)$, and suppose y to be increased by x, we then have (Art. 24),

$$\Delta u = f(y+x) - f(y) = Ax + m_2x^2, \quad . \quad . \quad (1)$$

or

$$f(y+x) = f(y) + Ax + m_2x^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where A $(= m_1 = f'(y))$ is a constant with respect to x, and m_2x^2 is the acceleration of u (Art. 25), and this is what we wish now to determine.

Since m_2 is of such a character that m_2x vanishes with x, we will assume (see footnote, p. 10)

$$m_2 = B + Cx + Dx^2 + \ldots Lx^{n-3}, \quad . \quad . \quad . \quad . \quad (3)$$

where $B, C, D, \ldots L$ are independent of x, and x has such a value as to render the series (if infinite) convergent. Substituting in (2), we have

$$f(y+x) = f(y) + Ax + Bx^2 + Cx^3 + \ldots Lx^{n-1}. \quad . \quad (4)$$

Differentiating (4) successively with respect to x, we have

$$f'(y+x) = A + 2Bx + 3Cx^2 + \dots (n-1)Lx^{n-2}; \quad (5)$$
$$f''(y+x) = 2B + 6Cx + \dots (n-1)(n-2)Lx^{n-3}; \quad (6)$$
$$f'''(y+x) = 6C + \dots (n-1)(n-2)(n-3)Lx^{n-4}; \quad (7)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots;$$
$$f^{n-1}(y+x) = \underline{|n-1}\, L. \quad \dots\dots\dots \quad (8)$$

These equations, (5), (6), (7), etc., are true for any value of x which renders equation (4) convergent (Art. 119); therefore they are true when $x = 0$, which gives

$$f'(y) = A, \qquad \therefore A = f'(y);$$
$$f''(y) = 2B, \qquad \therefore B = \frac{1}{\underline{|2}} f''(y);$$
$$f'''(y) = 6C, \qquad \therefore C = \frac{1}{\underline{|3}} f'''(y);$$
$$\dots\dots\dots\dots\dots\dots\dots;$$
$$f^{n-1}(y) = \underline{|n-1}\, L, \quad \therefore L = \frac{1}{\underline{|n-1}} f^{n-1}(y).$$

Substituting these values for A, B, C, ... L in (4), we have

$$\left.\begin{aligned} f(y+x) = f(y) + f'(y)x + f''(y)\frac{x^2}{\underline{|2}} + f'''(y)\frac{x^3}{\underline{|3}} \\ + \dots f^{n-1}(y)\frac{x^{n-1}}{\underline{|n-1}} + \dots \end{aligned}\right\} \quad \text{(A)}$$

This is the formula required, which was first published in 1715 by Dr. Brook Taylor, from whom it takes its name.

The preceding is not a rigorous demonstration of Taylor's formula, inasmuch as the possibility of development in the proposed form is assumed. A rigorous proof, including the form of the remainder, has been inserted in the Appendix, A_5, to be used or not, according as the teacher or student may desire.

122. COR. I. **To determine for what values of x the series is convergent.**

The nth and $(n+1)$th terms of the series are evidently

$$f^{n-1}(y)\frac{x^{n-1}}{\lfloor n-1} \quad \text{and} \quad f^{n}(y)\frac{x^{n}}{\lfloor n}.$$

Therefore (Art. 117) the series is convergent for any value of x numerically less than

$$\frac{f^{n-1}(y)}{\lfloor n-1} \div \frac{f^{n}(y)}{\lfloor n}, \quad \text{or} \quad n\left(\frac{f^{n-1}(y)}{f^{n}(y)}\right), \text{ when } n=\infty.$$

Hence, if $\dfrac{f^{n-1}(y)}{f^{n}(y)}$ is not zero when $n=\infty$, the series is convergent for all finite values of x; and if $n\dfrac{f^{n-1}(y)}{f^{n}(y)}=0$, when $n=\infty$, the series is divergent for all values of x except 0.

In deducing Taylor's formula we have supposed all the functions that occur to be continuous. Hence the formula is inapplicable, or "fails," if the function, or any of its differential coefficients, be infinite for values of the variable lying between the limits for which the development holds.

123. To develop $(y+x)^m$.

Here $\qquad f(y+x)=(y+x)^m.$

Make $x=0$, $\qquad f(y)=y^m.$

Differentiate, etc.,

$$\begin{aligned} f'(y) &= m\,y^{m-1}, \\ f''(y) &= m(m-1)y^{m-2}, \\ f'''(y) &= m(m-1)(m-2)y^{m-3}, \\ \text{etc.} & \qquad \text{etc.} \end{aligned}$$

Substituting these values in (A), we have

$$(y+x)^m = y^m + mx\,y^{m-1} + \frac{m(m-1)}{\lfloor 2}x^2y^{m-2} + \frac{m(m-1)(m-2)}{\lfloor 3}x^3y^{m-3}, \text{ etc.,} \qquad \text{(B)}$$

which is the **Binomial Theorem.**

Cor. I. Let us determine for what values of x the equation is true, supposing m negative or fractional.

$$f^{n-1}(y) = m(m-1)\ldots(m-n+2)y^{m-n+1},$$

$$f^{n}(y) = m(m-1)\ldots(m-n+1)y^{m-n};$$

$$\therefore \text{ (Art. 122)}, \qquad n\frac{f^{n-1}(y)}{f^{n}(y)} = \frac{ny}{m-n+1},$$

which $= -y$ when $n = \infty$.

Therefore formula (B) is true when x is numerically less than y.

Cor. II. Making y in (B) equal to 1, we have

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{\underline{|2}}x^2 + \frac{m(m-1)(m-2)}{\underline{|3}}x^3 + \text{etc.}, \qquad \text{(C)}$$

in which $-1 < x < +1$.

124. To develop $\sin(y+x)$.

Here $\qquad f(y+x) = \sin(y+x)$.

Making $x = 0$, and differentiating, we have

$$f(y) = \sin y; \qquad f'(y) = \cos y; \quad f''(y) = -\sin y;$$

$$f'''(y) = -\cos y; \quad f^{\text{iv}}(y) = \sin y; \quad \text{etc.}$$

Substituting these values in (A), we have

$$\left.\begin{aligned} \sin(y+x) = \sin y\left(1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \text{etc.}\right) \\ + \cos y\left(x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \text{etc.}\right) \end{aligned}\right\} \quad \text{(D)}$$

Cor. I. In (D) by making $y = 0$, remembering that $\sin 0 = 0$ and $\cos 0 = 1$, we have

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \text{etc.} \quad \ldots \quad \text{(E)}$$

Cor. II. Differentiating (E), we have

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \text{etc.} \quad . \quad . \quad . \quad \text{(F)}$$

Cor. III. For the quantities within the parentheses in (D), substituting their values from (E) and (F), we have

$$\sin (y + x) = \sin y \cos x + \cos y \sin x. \quad . \quad . \quad . \quad \text{(G)}$$

Cor. IV. Differentiating (G), regarding x as constant and y as variable, we have

$$\cos (y + x) = \cos y \cos x - \sin y \sin x. \quad . \quad . \quad . \quad \text{(H)}$$

125. To develop $\log (y + x)$.

$f(y + x) = \log (y + x)$; making $x = 0$, and differentiating, we have

$$f(y) = \log (y); \quad f'(y) = \frac{1}{y}; \quad f''(y) = -\frac{1}{y^2};$$

$$f'''(y) = \frac{2}{y^3}; \qquad f^{iv}(y) = -\frac{\lfloor 3}{y^4}, \text{ etc.}$$

Substituting in (A), we have

$$\log (y + x) = \log (y) + \frac{x}{y} - \frac{x^2}{2y^2} + \frac{x^3}{3y^3} - \text{etc.}, \quad . \quad . \quad \text{(I)}$$

which is the logarithmic series.

Cor. I. The nth and $(n + 1)$th terms of (I) are, omitting the signs, $\frac{x^{n-1}}{(n-1)y^{n-1}}$, and $\frac{x^n}{n y^n}$; hence, Art. 117, $\frac{a_{n-1}}{a_n} = \frac{ny}{n-1}$, which $= y$ when $n = \infty$. Therefore formula (I) is true when x is numerically less than y.

Cor. II. In (I), by making $y = 1$, we have

$$\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{etc.}, \quad . \quad . \quad \text{(K)}$$

which is true for all values of x numerically less than 1.

126. Maclaurin's Formula is a formula for developing a function of a single variable, as $y = a^x$, $y = \log(1+x)$, $y = (a+x)^n$.

It may be derived from (A) by making $y = 0$, which gives

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{\underline{|2}} + f'''(0)\frac{x^3}{\underline{|3}}$$

$$+ \ldots f^{n-1}(0)\frac{x^{n-1}}{\underline{|n-1}} + \ldots, \qquad \text{(L)}$$

in which $f(0), f'(0), f'''(0)$, etc., represent the values which $f(x)$ and its successive derivatives assume when $x = 0$.

COR. I. Formula (L) is true for all values of x numerically less than $\frac{nf^{n-1}(0)}{f^n(0)}$, when $n = \infty$.

127. To develop a^x.

$$\begin{array}{llll}
\text{Here} & f(x) = a^x, & \therefore f(0) & = a^0 = 1; \\
& f'(x) = a^x \log a, & \therefore f'(0) & = \log a; \\
& f''(x) = a^x \log^2 a, & \therefore f''(0) & = \log^2 a; \\
& f'''(x) = a^x \log^3 a, & \therefore f'''(0) & = \log^3 a; \\
& \text{etc.} & \text{etc.} &
\end{array}$$

Substituting these values in (L), we have

$$a^x = 1 + \log ax + \log^2 a\frac{x^2}{\underline{|2}} + \log^3 a\frac{x^3}{\underline{|3}} + \log^4 a\frac{x^4}{\underline{|4}}, \quad . \qquad \text{(M)}$$

which is called the **Exponential Series.**

COR. I. This series is convergent for all finite values of x, since $f^{n-1}(0) \div f^n(0)$ is obviously finite and different from zero for all values of n.

COR. II. Making $a = e$, remembering that $\log e = 1$, we have

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \text{etc.} \quad . \quad . \quad . \qquad \text{(N)}$$

COR. III. Making $x = 1$, we have

$$e = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \frac{1}{\lfloor 5} + \text{etc.} \quad . \quad . \quad (P)$$

Hence $e = 2.718281 +$.

128. Find the development of $\tan^{-1} x$.

In the applications of Maclaurin's formula the labor of finding the successive derivatives can often be lessened by taking the development of the first derivative, as follows:

$$\begin{array}{ll}
f(x) = \tan^{-1} x, & \therefore f(0) = \tan^{-1} 0 = 0; \\
f'(x) = \dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \text{etc.}, & \therefore f'(0) = 1; \\
f''(x) = -2x + 4x^3 - 6x^5 + \text{etc.}, & \therefore f''(0) = 0; \\
f'''(x) = -2 + 3 \cdot 4 \cdot x^2 - 5 \cdot 6x^4 + \text{etc.}, & \therefore f'''(0) = -2; \\
f^{iv}(x) = 2 \cdot 3 \cdot 4x - 4 \cdot 5 \cdot 6x^3 + \text{etc.}, & \therefore f^{iv}(0) = 0; \\
f^{v}(x) = \lfloor 4 - 3 \cdot 4 \cdot 5 \cdot 6x^2 + \text{etc.}, & \therefore f^{v}(0) = \lfloor 4; \\
f^{vi}(x) = -2 \cdot 3 \cdot 4 \cdot 5 \cdot 6x + \text{etc.}, & \therefore f^{vi}(0) = 0; \\
f^{vii}(x) = -\lfloor 6 + \text{etc.}, & \therefore f^{vii}(0) = -\lfloor 6; \\
\text{etc.}, & \text{etc.}
\end{array}$$

Substituting in (L), we have

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{etc.} \quad . \quad . \quad . \quad (Q)$$

EXAMPLES.

Develop the following:

1. $\sqrt{1 + x^2}$. $\qquad 1 + \dfrac{x^2}{2} - \dfrac{x^4}{8} + \dfrac{x^6}{16} - \dfrac{5x^8}{128} + \text{etc.}$

Put $x^2 = y$, and develop; then replace y by its value.

2. $(a + x)^{-3}$. $\qquad a^{-3} - 3a^{-4}x + 6a^{-5}x^2 - 10a^{-6}x^3 + \text{etc.}$

3. $(1 + x)^{\frac{1}{3}}$. $\qquad 1 + \dfrac{x}{3} - \dfrac{x^2}{9} + \dfrac{5x^3}{81} - \text{etc.}$

4. $e^{\sin x}$. $\qquad 1 + x + \frac{x^2}{2} - \frac{x^4}{2.4} + \frac{x^5}{3.5} - \frac{x^6}{2.4.5.6} + \text{etc.}$

5. $e^{\cos x}$. $\qquad e\left(1 - \frac{x^2}{2} + \frac{4x^4}{\underline{|4}} - \frac{31x^6}{\underline{|6}} + \text{etc.}\right)$.

6. $\tan x$. $\qquad x + \frac{x^3}{3} + \frac{2x^5}{15} + \text{etc.}$

7. $\sec x$. $\qquad 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \text{etc.}$

8. $\log(1 + \sin x)$. $\qquad x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \text{etc.}$

9. $\log(1 + e^x)$. $\qquad \log 2 + \frac{x}{2} + \frac{x^2}{8} - \text{etc.}$

10. $e^{x \sin x}$. $\qquad 1 + x^2 + \frac{x^4}{3} + \text{etc.}$

11. $e^x \sec x$. $\qquad 1 + x + x^2 + \frac{2x^3}{3} + \text{etc.}$

12. $\log(1 - x + x^2)$. $\qquad - x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4} - \text{etc.}$

Put $-x + x^2 = y$, and develop; then replace y by its value.

In the two following examples put y for x^2, develop, and replace y by its value.

13. $\sqrt{1 - x^2}$. $\qquad 1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128} - \text{etc.}$

14. $\frac{1}{\sqrt{1 + x^2}}$. $\qquad 1 - \frac{x^2}{2} + \frac{3x^4}{2.4} - \frac{3.5x^6}{2.4.6} + \text{etc.}$

129. To find the value of π.

We find by development that

$$\sin^{-1} x = x + \frac{x^3}{2.3} + \frac{1.3x^5}{2.4.5} + \frac{1.3.5x^7}{2.4.6.7} + \text{etc.},$$

where x lies between -1 and $+1$.

Making $x = \frac{1}{2}$, remembering that $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$, we have

$$\frac{\pi}{6} = \frac{1}{2}\left(1 + \frac{1}{24} + \frac{3}{640} + \frac{5}{7168} +\right),$$

or $\qquad \pi = 3.141592 + \ldots$

130. To compute Natural logarithms.

We have found

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}, \quad . \quad (1)$$

where x is numerically less than 1.

We now proceed to modify this series so that it shall be true and convergent for larger values of x.

Substituting $-x$ for x in (1), we have

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \text{etc.} \quad . \quad (2)$$

Subtracting (2) from (1), we have

$$\log(1+x) - \log(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \text{etc.}\right). \quad . \quad (3)$$

In (3) make $x = \frac{1}{2z+1}$, where z may have any positive value; then

$$\frac{1+x}{1-x} = \frac{z+1}{z}$$

and $\qquad \log(1+x) - \log(1-x) = \log(z+1) - \log(z)$,

we have $\qquad \log(z+1) = \log(z)$

$$+ 2\left[\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \text{etc.}\right] \quad (4)$$

By this series we can compute the Natural logarithm of any number $(z+1)$ when we know the logarithm of the number (z) less by unity.

Making $z = 1$, remembering that $\log(1) = 0$, we have

$$\log(2) = 2\left[\frac{1}{3} + \frac{1}{3 . 3^3} + \frac{1}{5 . 3^5} + \frac{1}{7 . 3^7} + \text{etc.}\right].$$

Taking six terms of this series, we have

$$\log 2 = .693147 + \ldots$$

Putting $z = 2$ in (4), we have

$$\log(3) = \log 2 + 2\left(\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \frac{1}{7 \cdot 5^7} + \text{etc.}\right)$$

$$= 1.098612 +$$

$$\log(4) = 2 \log 2 = 1.386294 +.$$

Putting $z = 4$, we have

$$\log(5) = \log 4 + 2\left(\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} + \frac{1}{7 \cdot 9^7} + \text{etc.}\right)$$

$$= 1.609437 +.$$

$$\log 10 = \log 2 + \log 5 = 2.302585 +.$$

In this way we can compute the natural logarithms of all numbers. It is not necessary to use the formula in finding the logarithms of composite members, for they can be found by simply adding the logarithms of their factors. Thus $\log 15 = \log 3 + \log 5$.

131. To compute common logarithms.

The modulus of the common system is $m = \log_{10} e$ (Art. 79). Hence $10^m = e$, $\therefore \log(10^m) = \log e$, or $m \log 10 = 1$.

$$\therefore m = \frac{1}{\log 10} = \frac{1}{2.302585} = .434294 +.$$

Let $\qquad \log_{10} v = n \quad \text{and} \quad \log v = n'$;

then $\qquad 10^n = v \quad \text{and} \quad e^{n'} = v$;

$\therefore \qquad 10^n = e^{n'}, \quad \log(10^n) = \log e^{n'}$, or

$$n \log 10 = n'. \qquad \therefore n = (n')\, .434294 +.$$

Hence, to find the common logarithm of any number we multiply the natural logarithm of that number by the modulus of the common system.

INDETERMINATE FORMS.

132. In algebra $\frac{0}{0}$ is called a symbol of indetermination, since any number whatever may assume this form.

Thus, $n \times 0 = 0$: divide both sides by 0 and we have $n = \frac{0}{0}$.

There are many fractions which assume the form of $\frac{0}{0}$ in consequence of one and the same supposition, which makes both numerator and denominator $= 0$. Such fractions are called **Vanishing Fractions,** and their values, which appear under the form of $\frac{0}{0}$, can generally be determined by the calculus.

Thus the fraction $\frac{x^3 - a^3}{x^2 - a^2}$ becomes $\frac{0}{0}$ when $x = a$.

This form arises from the existence of a factor $(x - a)$ common to both numerator and denominator, which factor becomes 0 under the particular supposition. Dividing both terms by this factor, we have

$$\frac{x^3 - a^3}{x^2 - a^2} = \frac{x^2 + ax + a^2}{x + a}, \text{ which } = \frac{3a}{2} \text{ when } x = a.$$

133. To evaluate a fraction that takes the form of $\frac{0}{0}$

Let u and v be functions of x such that when $x = a$, $u = 0$, and $v = 0$.

Let u and v be estimated from the point where their values are 0, that is, from where $x = a$; then when $x\,(= a)$ is increased by h we shall have, *identically,*

$$\frac{u}{v} = \frac{\Delta u}{\Delta v}.$$

As h approaches 0, or x approaches a, the limit of $\frac{\Delta u}{\Delta v}$ *is equal to* $\frac{du}{dv}$ (Art. 27); therefore

$$\left.\frac{u}{v}\right]_a = \left.\frac{du}{dv}\right]_a,$$

which is read, when $x = a$, $\frac{u}{v}$ is equal to $\frac{du}{dv}$.

Applying this to the preceding example, we have

$$\left.\frac{x^3 - a^3}{x^2 - a^2}\right]_a = \left.\frac{d(x^3 - a^3)}{d(x^2 - a^2)}\right]_a = \left.\frac{3x^2}{2x}\right]_a = \frac{3a}{2}.$$

An easy deduction of the rule for reckoning forms like $\frac{0}{0}$ is obtained by the use of Taylor's formula, as follows:

Let $\phi(x)$ and $f(x)$ be two functions of x such that $f(x) = 0$ and $\phi(x) = 0$, when $x = a$, then we shall have $\frac{\phi(a)}{f(a)} = \frac{0}{0}$.

Evidently $$\frac{\phi(a)}{f(a)} = \underset{h \doteq 0}{\text{limit}} \left[\frac{\phi(a + h)}{f(a + h)}\right]$$

$$= \underset{h \doteq 0}{\text{limit}} \left[\frac{\phi(a) + \phi'(a)h + \frac{1}{2}\phi''(a)h^2 + \ldots}{f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + \ldots}\right] = \frac{\phi'(a)}{f'(a)} = \left.\frac{\phi(x)}{f(x)}\right]_a.$$

Cor. I. If $\phi'(a) = 0$ and $f'(a) = 0$, we obviously have

$$\frac{\phi''(a)}{f''(a)} = \left.\frac{\phi(x)}{f(x)}\right]_a;$$

and if $\phi''(a) = 0$, and $f''(a) = 0$, we have

$$\frac{\phi'''(a)}{f'''(a)} = \left.\frac{\phi(x)}{f(x)}\right]_a; \text{ and so on.}$$

Hence, Rule. *Substitute for the numerator and denominator, respectively, their first derivatives, or their second derivatives, and so on, till a fraction is obtained whose terms do*

not both become 0 when $x = a$; the values thus found will be the true value of the vanishing fraction.

EXAMPLES.

Compute the following:

1. $\left.\dfrac{x^5 - 1}{x^2 - 1}\right]_1.$ $\qquad \dfrac{5}{2}.$

2. $\left.\dfrac{x^2 - 16}{x^2 + x - 20}\right]_4.$ $\qquad \dfrac{8}{9}.$

3. $\left.\dfrac{5x^2 - 8x + 3}{7x^2 - 9x + 2}\right]_1.$ $\qquad \dfrac{2}{5}.$

4. $\left.\dfrac{x^3 - 3x + 2}{x^3 - x^2 - x + 1}\right]_1.$ $\qquad \dfrac{3}{2}.$

5. $\left.\dfrac{1 - \cos x}{\sin x}\right]_0.$ $\qquad 0.$

6. $\left.\dfrac{a - \sqrt{a^2 - x^2}}{x^2}\right]_0.$ $\qquad \dfrac{1}{2a}.$

7. $\left.\dfrac{\sin x}{x}\right]_0.$ $\qquad 1.$

8. $\left.\dfrac{x - \sin x}{x^3}\right]_0.$ $\qquad \dfrac{1}{6}.$

9. $\left.\dfrac{\sin 3x}{x - \frac{3}{2}\sin 2x}\right]_0.$ $\qquad -\dfrac{3}{2}.$

10. $\left.\dfrac{e^x - e^{-x}}{\sin x}\right]_0.$ $\qquad 2.$

11. $\left.\dfrac{\log \sin x}{(\pi - 2x)^2}\right]_{\frac{\pi}{2}}.$ $\qquad -\dfrac{1}{8}.$

12. $\left.\dfrac{\sec^2 x - 2\tan x}{1 + \cos 4x}\right]_{\frac{\pi}{4}}.$ $\qquad \dfrac{1}{2}.$

13. $\left.\dfrac{\sin x - x\cos x}{x - \sin x}\right]_0.$ $\qquad 2.$

14. $\left.\dfrac{(x - 2)e^x + x + 2}{(e^x - 1)^3}\right]_0.$ $\qquad \dfrac{1}{6}.$

There are other indeterminate forms besides $\frac{0}{0}$, such as $\frac{\infty}{\infty}$, $\infty \times 0$, $\infty - \infty$, 0^0, ∞^0, 1^∞, which will be considered in succession.

134. To evaluate a fraction that takes the form of $\frac{\infty}{\infty}$.

Let u and v be functions of x such that $u = \infty$ and $v = \infty$ when $x = a$; then for the same value of x, $\frac{1}{u} = 0$ and $\frac{1}{v} = 0$. Hence

$$\frac{u}{v} = \frac{\frac{1}{v}}{\frac{1}{u}} = \frac{0}{0}, \text{ when } x = a.$$

$$\therefore \text{ Art. 133,} \qquad \frac{u}{v} = \frac{d\left(\frac{1}{v}\right)}{d\left(\frac{1}{u}\right)} = \frac{u^2 dv}{v^2 du}, \text{ when } x = a. \quad . \quad . \quad . \quad (1)$$

Dividing (1) by $\frac{u}{v}$, we obtain

$$1 = \frac{u dv}{v du}, \quad \text{or} \quad \left.\frac{u}{v}\right]_a = \left.\frac{du}{dv}\right]_a. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now (2) is derived from (1) by dividing by $\frac{u}{v}$; hence, if $\frac{u}{v}$ is finite, (2) is true for all finite values of $\frac{u}{v}$; and if $\frac{u}{v} = 0$ or ∞, it may be shown that (2) is true in these cases also.

Suppose $\frac{u}{v} = 0$ when $x = a$, and k a finite quantity,

$$\text{then} \qquad \frac{u}{v} + k = \frac{u + kv}{v} = k.$$

To this last fraction (2) evidently applies, hence

$$\frac{u + kv}{v} = \frac{du + kdv}{dv}, \quad \text{or} \quad \frac{u}{v} + k = \frac{du}{dv} + k;$$

$$\text{that is,} \qquad \frac{u}{v} = \frac{du}{dv}, \quad \text{when } x = a.$$

If $\frac{u}{v} = \infty$, then $\frac{v}{u} = 0$, and we have the preceding case.

Therefore the form $\frac{\infty}{\infty}$ is to be evaluated in the same way as the form $\frac{0}{0}$.

EXAMPLES.

Find the values of the following:

1. $\left.\frac{x}{\log x}\right]_\infty$. $\qquad \infty$.

$$= \left.\frac{d(x)}{d(\log x)}\right]_\infty = \left.\frac{1}{\frac{1}{x}}\right]_\infty = x\Big]_\infty.$$

2. $\left.\frac{ax^2 + b}{cx^2 + d}\right]_\infty$. $\qquad \frac{a}{c}$.

3. $\left.\frac{a^x - 1}{x}\right]_\infty$. $\qquad \infty$.

4. $\left.\frac{\log\left(x - \frac{\pi}{2}\right)}{\tan x}\right]_{\frac{\pi}{2}}$. $\qquad 0$.

5. $\left.\frac{\log x}{\cot x}\right]_0$. $\qquad 0$.

6. $\left.\frac{\log \cot x}{\operatorname{cosec} x}\right]_0$. $\qquad 0$.

7. $\left.\frac{\log \tan 2x}{\log \tan x}\right]_{\frac{\pi}{2}}$. $\qquad -1$.

8. $\left.\frac{\tan\left[\frac{\pi}{4}(x+1)\right]}{\tan\frac{\pi x}{2}}\right]_1$. $\qquad 2$.

9. $\left.\frac{\log\cos\left(\frac{1}{2}\pi x\right)}{\log(1-x)}\right]_1$. $\qquad 1$.

135. To evaluate a function that takes the form of $0 \times \infty$ or $\infty - \infty$.

Functions of this kind can be transformed so as to assume the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and then be evaluated by the previous methods.

EXAMPLES.

Find the following:

1. $\sec 3x \cos 5x \big]_{x=\frac{\pi}{2}}$. $\quad -\frac{5}{3}$.

This takes the form of $\infty \times 0$; but $\sec 3x \cos 5x = \frac{\cos 5x}{\cos 3x}$, which takes the form of $\frac{0}{0}$.

2. $\sec x - \tan x \big]_{x=\frac{\pi}{2}}$. $\quad 0$.

This takes the form of $\infty - \infty$; but $\sec x - \tan x = \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \frac{1 - \sin x}{\cos x}$, which takes the form of $\frac{0}{0}$.

3. $\left.\frac{1}{\log x} - \frac{1}{x-1}\right]_1$. $\quad \frac{1}{2}$.

4. $\left.\operatorname{cosec}^2 x - \frac{1}{x^2}\right]_0$. $\quad \frac{1}{3}$.

5. $\left.\frac{e}{e^x - e} - \frac{1}{x-1}\right]_1$. $\quad -\frac{1}{2}$.

6. $(1 - \tan x) \sec 2x\big]_{\frac{\pi}{4}}$. $\quad 1$.

7. $\left.\frac{x^2 - a^2}{a^2} \tan \frac{\pi x}{2a}\right]_a$. $\quad -\frac{4}{\pi}$.

8. $\left.x \sin \frac{a}{x}\right]_\infty$. $\quad a$.

9. $x^m \log^n x\big]_0$. (m and n being $+$.) $\quad 0$.

10. $(1 - x) \tan (\frac{1}{2}\pi x)\big]_1$. $\quad \frac{2}{\pi}$.

136. To evaluate a function that takes the form of 0^0, ∞^0, or 1^∞.

Take the logarithm of the given function, which will assume the form of $0 \times \infty$, and can be evaluated by Art. 135. From this the value of the function can be found.

EXAMPLES.

Find the following:

1. $\left.\left(1+\frac{a}{x}\right)^x\right]_\infty$. $\qquad e^a$.

This takes the form of 1^∞. Making $y = \left(1+\frac{a}{x}\right)^x$, we have $\log y = x \log\left(1+\frac{a}{x}\right)$. The value of $\left. x \log\left(1+\frac{a}{x}\right)\right]_\infty$ is found by Art. 135 to be a. Hence, when $x = \infty$, $\log y = a$; $\therefore y = e^a$.

2. $\left.\sqrt[x]{x}\right]_\infty$, or $\left. x^{\frac{1}{x}}\right]_\infty$. $\qquad 1$.

This takes the form of ∞^0; the log of $x^{\frac{1}{x}}$ is $\frac{1}{x}\log x$, the value of which, when $x = \infty$, is 0; hence $\left. x^{\frac{1}{x}}\right]_\infty = 1$.

3. $\left.(\sin x)^{\tan x}\right]_{\frac{\pi}{2}}$. $\qquad 1$.

4. $\left.(\cot x)^{\sin x}\right]_0$. $\qquad 1$.

5. $\left.(e^x+1)^{\frac{1}{x}}\right]_\infty$. $\qquad e$.

6. $\left.\left(\tan \frac{\pi x}{4}\right)^{\tan \frac{\pi x}{2}}\right]_1$. $\qquad \frac{1}{e}$.

7. $\left.(\cot x)^{\frac{1}{\log x}}\right]_0$. $\qquad \frac{1}{e}$.

8. $\left.\sqrt[x]{\log(e+x)}\right]_0$. $\qquad e^{\frac{1}{e}}$.

9. $\left.\sqrt[x]{e^x+x}\right]_0$. $\qquad e^2$.

10. $\left.\sqrt[x^2]{\cos 2x}\right]_0$. $\qquad \frac{1}{e^2}$.

11. $\sqrt[x]{\dfrac{\log x}{x}}\Bigg]_{\infty}$. **1.**

12. $(\cos mx)^{\frac{n}{x^2}}\Big]_0$. $e^{-\frac{1}{2}nm^2}$.

137. In implicit functions, as $f(x, y) = 0$, the derivative $\dfrac{dy}{dx}$ can be evaluated by the previous methods when it assumes an indeterminate form for particular values of x and y.

EXAMPLES.

1. Find the slope of $x^4 - a^2xy + b^2y^2$ at the point (0, 0).

Here $$\frac{dy}{dx} = \frac{4x^3 - a^2y}{a^2x - 2b^2y} = \frac{0}{0} \text{ when } x = y = 0.$$

$$\therefore \quad \frac{dy}{dy} = \frac{12x^2 - a^2\dfrac{dy}{dx}}{a^2 - 2b^2\dfrac{dy}{dx}} = \frac{-a^2\dfrac{dy}{dx}}{a^2 - 2b^2\dfrac{dy}{dx}}, \text{ when } x = y = 0;$$

that is, $$\frac{dy}{dx} = \frac{-a^2\dfrac{dy}{dx}}{a^2 - 2b^2\dfrac{dy}{dx}}, \quad \text{or} \quad 2b^2\left(\frac{dy}{dx}\right)^2 - a^2\left(\frac{dy}{dx}\right) = a^2\frac{dy}{dx};$$

$$\therefore \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{a^2}{b^2}.$$

2. Find the slopes of $x^3 - 3axy + y^3 = 0$ at (0, 0).
   0 and ∞.

3. Find the slopes of $x^4 + ax^2y - ay^3 = 0$ at (0, 0).
   0 and ± 1.

4. Find the slopes of $y^2 = x(x + a)^2$ at $(-a, 0)$. $\pm\sqrt{-a}$.

5. Find the slopes of $x^4 + 2ax^2y - ay^3 = 0$ at (0, 0).
   0 and $\pm\sqrt{2}$.

CHAPTER VI.

MAXIMA AND MINIMA.

DEFINITIONS AND PRINCIPLES.

138. A **Maximum Value** of a function is a value that is greater than its immediately preceding and succeeding values, and a **Minimum Value** is one that is less than its immediately preceding and succeeding values.

Thus, while x increases continuously, if $f(x)$ increases up to a certain value, say $f(a)$, and then decreases, $f(a)$ is a maximum value of $f(x)$; and if, while x increases, $f(x)$ decreases to a certain value, say $f(b)$, and then increases, $f(b)$ is a minimum value of $f(x)$.

For example, $\sin x$ increases as x increases till the latter reaches $90°$, after which $\sin x$ decreases as x goes on increasing; that is, $\sin 90°$ is a maximum value of $\sin x$.

Again, if x increases continuously from 0 to 5, $f(x) = x^2 - 6x + 10$ will decrease until x becomes 3 and then it will increase; hence $f(3) = 1$ is a minimum value of $f(x)$, or $x^2 - 6x + 10$.

Let the student substitute 1, 2, 3, . . . 10, successively, for x in $f(x) = x^3 - 18x^2 + 96x - 20$, and thus show that $f(4)$ is a maximum and $f(8)$ is a minimum.

139. *Any value of x that renders $f(x)$ a maximum or a minimum is a root of the equation $f'(x) = 0$ or ∞, if $f(x)$ and $f'(x)$ vary continuously with x.*

For, if we conceive x as always increasing, $f(x)$ changes from an increasing to a decreasing function as it passes through a

maximum value, say $f(a)$, and from a decreasing to an increasing function as it passes through a minimum value, say $f(b)$. Consequently $f'(x)$ must change sign as x passes through a or b, Art. 25. But $f'(x)$ can change sign only by passing through 0 or ∞. Therefore $f'(a)$ or $f'(b) = 0$ or ∞; that is, a and b are roots* of $f'(x) = 0$ or ∞.

To illustrate the preceding principles and definitions graphically, let $y = f(x)$ be the equation of the curve Am; then $f'(x) =$ the slope of the curve at the point P or (x, y), Art. 48. As x $(= OB)$ increases, the point P will move from A along the curve to the right, and y or $f(x)$ will increase till it becomes aa', and then decrease till it becomes bb', and then increase, etc. Therefore aa'_1, cc_1', ee' are maximum, and bb', dd' are minimum, values of $f(x)$. The slope of the curve, $f'(x)$, is evidently positive before, and negative after, each maximum value of $f(x)$; and negative before, and positive after, each minimum value of $f(x)$. Moreover, at the points where $f(x)$ is a maximum or a minimum, the curve is either parallel or perpendicular to the axis of x, and therefore $f'(x) = 0$ or ∞.

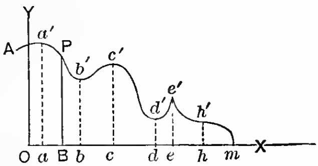

FIG. 19.

The converse of this theorem is not always true; that is, any root of $f'(x) = 0$ or ∞ does not necessarily render $f(x)$ a maximum or a minimum. It is our purpose now to determine which of the roots will render $f(x)$ a maximum and which a minimum.

140. *If the sign of $f'(x)$ undergoes no change as $f'(x)$ passes through 0 or ∞, the corresponding value of $f(x)$ will be neither a maximum nor a minimum.*

For, so long as the sign of $f'(x)$ undergoes no change, $f(x)$

---

* Here, and in what follows, the word *root* includes the real values of x which satisfy the equations $f'(x) = 0$ or ∞, whether $f'(x)$ be an algebraic or a transcendental function.

does not change from an increasing to a decreasing function, nor *vice versa.*

COR. I. If an even number of the roots of $f'(x) = 0$ or ∞ are equal to a, then $x = a$ will not render $f(x)$ a maximum or a minimum.

141. *If the sign of* $f'(x)$ *undergoes a change as* $f'(x)$ *passes through* 0 *or* ∞, *the corresponding value of* $f(x)$ *is a maximum or a minimum.*

For, if $f'(x)$ undergoes a change of sign, $f(x)$ necessarily passes from an increasing to a decreasing function, or *vice versa.*

COR. I. If an odd number of the roots of $f'(x) = 0$ or ∞ are equal to a, then $x = a$ will render $f(x)$ a maximum or a minimum.

COR. II. Therefore, omitting the equal roots of which the number is even, every real root of $f'(x) = 0$ or ∞ will render $f(x)$ either a maximum or a minimum. Now, of these let us find which will render $f(x)$ a maximum and which a minimum.

142. *Maxima and minima of a function occur alternately.*

For, suppose that $f(a)$ and $f(b)$ are maxima of $f(x)$, where $a < b$. Just after passing through $f(a)$, $f(x)$ is decreasing, and increasing just before it reaches $f(b)$; but in passing from a decreasing to an increasing state it must pass through a minimum; hence there is one minimum between every two consecutive maxima.

COR. I. Denote the roots of $f'(x) = 0$ and $f'(x) = \infty$ which will render $f(x)$ a maximum or a minimum by a_1, a_2, a_3, a_4, etc., in ascending order of algebraic magnitude. Then, if $f(x)$ is an increasing function for all values of x less than a_1, that is, if $f'(a_1 - h)$ is positive, h being ever so small, $f(a_1)$, $f(a_3)$, $f(a_5)$, etc., are maxima, and $f(a_2)$, $f(a_4)$, etc., are minima; and if $f(x)$ is a decreasing function for the same values of x, that is, if $f'(a_1 - h)$ is negative, the maxima and minima will be interchanged.

RULES FOR FINDING MAXIMUM AND MINIMUM VALUES OF FUNCTIONS.

143. The preceding principles indicate the following rule for finding the values of x which will render any function as $f(x)$ a maximum or minimum:

Differentiate the function $f(x)$; make $f'(x) = 0$ and $f'(x) = \infty$; find the real roots of both equations, and arrange all of them in order of algebraic magnitude, as a_1, a_2, a_3, etc., omitting the equal roots when there are an even number of them; substitute $-\infty$ or $a_1 - h$, h being very small, for x in $f'(x)$, and (1) if the result is $+$, a_1, a_3, etc., will each render $f(x)$ a maximum, and a_2, a_4, etc., will each render $f(x)$ a minimum; (2) if the result is $-$, $f(a_1)$, $f(a_3)$, etc., will be minima, and $f(a_2)$, $f(a_4)$, etc., will be maxima.

144. The preceding rule requires that all the real roots shall be found; it is sometimes desirable to know independently whether any particular root as a' will render $f(x)$ a maximum or minimum. This may be done thus:

I. Substitute $a' - h$ and $a' + h$ for x in $f'(x)$, h being a small quantity, and (1) if $f'(a' - h)$ is $+$, and $f'(a' + h)$ is $-$, $f(a')$ will be a maximum; (2) if $f'(a' - h)$ is $-$ and $f'(a' + h)$ is $+$, $f(a')$ will be a minimum, and if $f'(a' - h)$ and $f'(a' + h)$ have the same sign, $f(a')$ will be neither a maximum nor a minimum.

145. II. Developing $f(x - h)$ and $f(x + h)$ by Taylor's formula, substituting a' for x, transposing $f(a')$, and remembering that $f'(a') = 0$, we have

$$f(a' - h) - f(a') = f''(a')\frac{h^2}{\underline{|2}} - f'''(a')\frac{h^3}{\underline{|3}} + f^{\text{iv}}(a')\frac{h^4}{\underline{|4}} - \quad (1)$$

and

$$f(a' + h) - f(a') = f''(a')\frac{h^2}{\underline{|2}} + f'''(a')\frac{h^3}{\underline{|3}} + f^{\text{iv}}(a')\frac{h^4}{\underline{|4}} + . \quad (2)$$

If h be taken very small, the sign of the second member of either (1) or (2) will be the same as the sign of its first term. Hence, if $f''(a')$ is negative, $f(a')$ is greater than both $f(a'-h)$ and $f(a'+h)$, and therefore a maximum; while if $f''(a')$ is positive, $f(a')$ is less than both $f(a'-h)$ and $f(a'+h)$, and therefore a minimum. If $f''(a')=0$, and $f'''(a')$ is not 0, $f(a')$ is neither greater than both $f(a'-h)$ and $f(a'+h)$ nor less than both, and is therefore neither a maximum nor a minimum. If $f'''(a')$ as well as $f''(a')$ is 0, then, as before, $f(a')$ will be a maximum or a minimum according as $f^{\text{iv}}(a')$ is negative or positive; and so on.

Hence if a' is a root of $f'(x)=0$ or ∞, substitute it for x in the successive derivatives of $f(x)$. If the first derivative that does not reduce to 0 is of an odd order, $f(a')$ is neither a maximum nor a minimum; but if the first derivative that does not reduce to 0 is of an even order, $f(a')$ is a maximum or a minimum, according as this derivative is negative or positive.

NOTE.—In many instances this rule is impracticable on account of the great labor involved in finding the successive derivatives.

146. The following principles are self-evident, and often serve to facilitate the solution of problems in maxima and minima:

(1) If c is positive, $f(x)$ and $c \times f(x)$ are maxima or minima for the same value of x; hence a constant positive factor or divisor may be rejected in finding this value of x.

(2) $\log f(x)$ and $f(x)$ are maxima or minima for the same value of x; hence log may be rejected.

(3) $c+f(x)$ and $f(x)$ are maxima or minima for the same value of x; hence the constant c may be rejected.

(4) If n is a positive whole number, $[f(x)]^n$ and $f(x)$ are maxima or minima for the same value of x; hence in $[f(x)]^{\frac{p}{q}}$ the denominator q may be rejected, or in $\sqrt[n]{f(x)}$ the radical may be removed.

EXAMPLES.

1. Find what values of x will render $x^3 - 3x^2 - 24x + 85$ a maximum or a minimum.

Here $f(x) = x^3 - 3x^2 - 24x + 85$, $f'(x) = 3x^2 - 6x - 24$, $f''(x) = 6x - 6$. Making $f'(x) = 0$, we have $3x^2 - 6x - 24 = 0$, the roots of which are $x = +4$, $x = -2$. Now to determine whether these values of x give maxima or minima values of $f(x)$, we substitute them for x in $f''(x)$.

Thus: $$f''(4) = 6 \times 4 - 6 = +18,$$

$$f''(-2) = 6 \times -2 - 6 = -18.$$

Hence, Art. 145, when $x = 4$, $f(x)$ is a minimum, and when $x = -2$, $f(x)$ is a maximum.

Let the student construct the locus of $y = x^3 - 3x^2 - 24x + 85$, and thus exhibit these results graphically.

2. Examine $f(x) = x^3 - 3x^2 + 3x + 7$ for maxima and minima.

Here $f'(x) = 3x^2 - 6x + 3$, $f''(x) = 6x - 6$, $f'''(x) = 6$. The roots of $3x^2 - 6x + 3 = 0$ are $x = 1$, $x = 1$. Substituting these values of x in $f''(x)$ and $f'''(x)$, we have $f''(1) = 0$, $f'''(1) = 6$. Therefore the function $f(x)$ has neither a maximum nor minimum value, which we also infer from the fact that the *two* roots of $f'(x) = 0$ are equal, Art. 140, Cor. I.

3. Find the maxima and minima of $x^5 - 5x^4 + 5x^3 - 1$.

Here $f'(x) = 5x^4 - 20x^3 + 15x^2$; $\therefore f'(x) = 0$ is $5x^4 - 20x^3 + 15x^2 = 0$, or $(x^2 - 4x + 3)x^2 = 0$; the four roots of which are 0, 0, 1, and 3. Rejecting the two equal roots, Art. 140, we have $a_1 = 1$, $a_2 = 3$, and since $f'(-\infty) = 5(-\infty)^2$ is $+$, the given function is a maximum when $x = 1$, and a minimum when $x = 3$.

Therefore $$f(1) = 0 \text{ is a maximum,}$$

and $$f(3) = -28 \text{ is a minimum.}$$

4. Examine $(x-1)^4(x+2)^3$ for maxima and minima.
Differentiating and reducing, we have

$$f'(x) = (x-1)^3(x+2)^2(7x+5)$$

The roots of $f'(x) = 0$ are those of $(x-1)^3 = 0$, $(x+2)^2 = 0$ and $7x+5=0$; hence there are three roots each equal to 1, two each equal to -2, and one equal to $-\frac{5}{7}$.

Rejecting the two equal roots, we have $a_1 = -\frac{5}{7}$ and $a_2 = 1$, and since $f'(-\infty) = (-\infty)^3(-\infty)^2(-7\infty)$ is $+$, $f(x)$ is a maximum when $x = -\frac{5}{7}$, and a minimum when $x = 1$.

5. Determine when $b + c(x-a)^{\frac{2}{3}}$ is a maximum or minimum.

By (3) of Art. 146 we may remove b, by (1) c, and by (4) 3; hence we have $f(x) = (x-a)^2$; $\therefore f'(x) = 2(x-a)$, and $x-a=0$, or $a_1 = a$; and since $f'(-\infty)$ is $-$, the given function is a minimum when $x = a$.

6. Find the maximum and minimum values of $f(x) = \dfrac{(x+3)^3}{(x+2)^2}$.

Here $$f'(x) = \frac{x(x+3)^2}{(x+2)^3}.$$

I. $f'(x) = 0$ gives $x(x+3)^2 = 0$, of which one root is 0 and the other two are -3 and -3.

II. $f'(x) = \infty$ gives $(x+2)^3 = 0$, the three roots of which is each -2.

Rejecting the *two* equal roots, we have $a_1 = -2$, $x_2 = 0$; and since $f'(-\infty)$ is $+$, $f(-2) = \infty$ is a maximum value of $f(x)$, and $f(0) = \frac{27}{4}$ is a minimum.

7. If the derivative of $f(x)$ is $f'(x) = x^2 - 10x + 21$, what values of x will render $f(x)$ a maximum or minimum.

The roots of $x^2 - 10x + 21 = 0$ are 3 and 7; substituting these for x in $f''(x) = 2x - 10$, we have $f''(3) = 6 - 10 = -4$, and $f''(7) = 14 - 10 = +4$; therefore $f(3)$ is a maximum and $f(7)$ is a minimum.

Find the values of x which will give maximum and minimum values of the following functions:

8. $u = x^2 - 8x + 12$. $x = 4$, min.

9. $u = x^3 - 3x^2 - 24x + 85$. $x = -2$, max. $x = 4$, min.

10. $u = x^3 - 3x^2 + 6x + 7$ Neither a max nor min.

11. $u = 2x^3 - 21x^2 + 36x - 20$. $x = 1$, max.. $x = 6$, min.

12. $u = (x - 9)^3(x - 8)^4$. $x = 8$, max., $x = 8\frac{4}{7}$, min.

13. $u = \dfrac{x^2 - 7x + 6}{x - 10}$ $x = 4$, max.; $x = 16$, min.

14. $u = \dfrac{(x + 2)^3}{(x - 3)^2}$. $x = 3$, max , $x = 13$, min.

15. $u = \dfrac{x^2 + 3}{x - 1}$. $x = -1$, max.; $x = 3$, min.

16. $u = \dfrac{1 - x + x^2}{1 + x - x^2}$. $x = \frac{1}{2}$, min.

17 $u = \dfrac{(a - x)^3}{a - 2x}$. $x = \dfrac{a}{4}$, min.

18. $u = \dfrac{9}{x} + \dfrac{4}{3 - x}$. $x = 9$, max.; $x = 1\frac{4}{5}$, min.

19 $u = \sin x + \cos x$. $x = \dfrac{\pi}{4}$, max.

20. $u = \sin x\,(1 + \cos x)$. $x = \dfrac{\pi}{3}$, max.

21. $u = \dfrac{\sin x}{1 + \tan x}$. $x = \dfrac{\pi}{4}$, max.

22. $u = \dfrac{x}{1 + x \tan x}$. $x = \cos x$, max.

23. $u = \left(1 + x^{\frac{2}{3}}\right)(7 - x)^2$. $x=1$, max.; $x=0$ and $x=7$, min.

24. If $x + y = n$, what is the greatest possible value of xy? $\frac{1}{4}n^2$.

Make $$u = xy = x(n - x) = nx - x^2.$$

25. If $y = mx + c$, find the least possible value of $\sqrt{x^2 + y^2}$ $\dfrac{c}{\sqrt{m^2 + 1}}$.

Make $$u = \sqrt{x^2 + y^2} = \sqrt{(1 + m^2)x^2 + 2mcx + c^2}.$$

26. A merchant bought a bolt of linen, paying as many cents for each yard as there were yards in the bolt, and sold it at 20 cts. per yard; required the greatest possible profit. \$1.00.

27. A club of x members has $x^3 - 12x^2 + 45x + 15$ dollars in its treasury; how much is that apiece if the amount is (1) a minimum? (2) A maximum? (1) \$13.00; (2) \$23.00.

28. Find the value of ϕ when $\sin\phi - \cos\phi$ is a maximum.

$$\phi = \cos^{-1} - \tfrac{1}{2}\sqrt{2} = 135°.$$

29. Find the fraction that exceeds its square by the greatest possible quantity. $\frac{1}{2}$.

30. Find the fraction that exceeds its nth power by the greatest possible quantity. $\left(\frac{1}{n}\right)^{\frac{1}{n-1}}$.

31. Find a number x such that its xth root shall be a maximum. $x = e = 2.7182 +$.

32. Find the altitude of the maximum rectangle inscribed in a given triangle.

Let ABC be the triangle and $HKFE$ the required maximum rectangle; let $b = AB$, $h = CD$, $x = DG$, and $u =$ area of $HKFE$; then

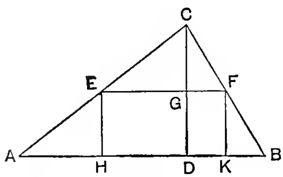

FIG. 20.

$$u = EF \times DG = (EF)x.$$

To express EF in terms of x, we have

$$CD : CG :: AB : EF,$$

or $$h : h - x :: b : EF;$$

$\therefore\ EF = \dfrac{b}{h}(h - x)$, and $u = \dfrac{b}{h}(hx - x^2)$, whose maximum value is required.

Dropping $\dfrac{b}{h}$, we have $f(x) = hx - x^2$, $\therefore f'(x) = h - 2x = 0$, whence $x = \dfrac{h}{2}$; that is, the altitude of the maximum rectangle is one half of the altitude of the triangle.

33. Find the altitude of the maximum rectangle inscribed in a given parabola.

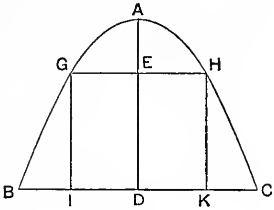

FIG. 21.

Let BAC be the parabola, and IH the rectangle; let $h = AD$, $x = AE$, $y = EH$, and $u =$ area of IH.

Then $$y^2 = 4ax,$$
and $$u = 2(ED)(EH) = 2(h - x)y$$
$$= 2(h - x)\sqrt{4a}\,x^{\frac{1}{2}}$$
$$= 4\sqrt{a}\left(hx^{\frac{1}{2}} - x^{\frac{3}{2}}\right).$$

Hence $f(x) = hx^{\frac{1}{2}} - x^{\frac{3}{2}}$, and $f'(x) = \frac{1}{2}hx^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} = 0$,

or $\dfrac{h}{\sqrt{x}} = 3\sqrt{x}$; whence $x = \frac{1}{3}h$; $\therefore DE = \frac{2}{3}h$.

34. Find the altitude of a maximum cylinder with respect to its volume that can be inscribed in a given right cone.

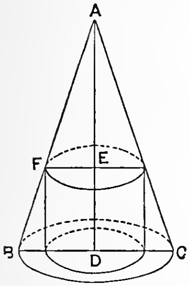

FIG. 22.

Let ED be the altitude of the cylinder inscribed in the cone DAC. Let $BD = b$, $AD = h$, $ED = x$, $EF = y$, and $v =$ volume of the cylinder; then
$$v = \pi(EF)^2 ED = \pi y^2 x.$$

To express y in terms of x, we have
$$AD : BD :: AE : EF, \text{ or } h : b :: h - x : y;$$
whence
$$y = \frac{b}{h}(h - x), \quad \text{and} \quad v = \pi\frac{b^2}{h^2}(h-x)^2 x.$$
$$\therefore f(x) = (h - x)^2 x = h^2 x - 2hx^2 + x^3,$$
which is to be a maximum.

$f'(x) = h^2 - 4hx + 3x^2 = 0$; whence $3x - h = 0$, or $x = \frac{1}{3}h$.

That is, the altitude of the cylinder is $\frac{1}{3}$ of the altitude of the cone.

35. Find the altitude of the cylinder in Fig. 22, if the cylinder is a maximum with respect to its lateral surface.

Denoting the lateral surface of the cylinder by S, we have $S = 2\pi(EF)ED = 2\pi yx = 2\pi\frac{b}{h}(h - x)x$, which is to be a maximum. Dropping the constant factor $2\pi\frac{b}{h}$, we have

$$f(x) = hx - x^2; \quad \therefore f'(x) = h - 2x = 0, \text{ or } x = \frac{h}{2}.$$

36. Find the dimensions of a cylindrical open-top vessel which has the least surface with a given capacity.

Let $x =$ the radius of the base, $y =$ the altitude, $s =$ the surface, and $c =$ the capacity.

Then $\quad c = \pi x^2 y \ . \ . \ . \ (1) \quad$ and $\quad s = \pi x^2 + 2\pi xy. \ . \ . \ . \ (2)$

From (1), $y = \frac{c}{\pi x^2}$; $\therefore s = \pi x^2 + \frac{2c}{x}$, which is to be a minimum.

$$\therefore f(x) = \pi x^2 + \frac{2c}{x}, \quad f'(x) = 2\pi x - \frac{2c}{x^2} = 0;$$

whence $x = \sqrt[3]{\frac{c}{\pi}}$, and $y = \sqrt[3]{\frac{c}{\pi}}$, which is obtained by substituting for x in $y = \frac{c}{\pi x^2}$.

37. A rectangle is inscribed in a circle whose radius is R; find the sides of the rectangle when it is a maximum (1) with respect to its area, (2) with respect to its perimeter.

Each side $= R\sqrt{2}$ in both cases.

38. The hypothenuse of a right triangle is h; find the ratio of the other sides when the triangle is a maximum (1) with respect to its area, (2) with respect to its perimeter.

Ratio $= 1$ in both cases.

39. A cylinder is inscribed in a sphere whose radius is R; find the radius of the cylinder when it is a maximum (1) with respect to its volume, (2) with respect to its convex surface.

(1) $\frac{1}{3}\sqrt{6}\,R$; (2) $\frac{1}{2}\sqrt{2}\,R$.

40. A cone is inscribed in a sphere whose radius is R; find the altitude of the cone when it is a maximum (1) with respect to its volume, (2) with respect to its convex surface.

(1) $\frac{4}{3}R$; (2) $\frac{4}{3}R$.

41. Find the maximum isosceles triangle with respect to its area that can be inscribed in a given circle.

An equilateral triangle.

42. Find the dimensions of a cone which has the greatest volume with a given amount of surface.

The slant height is three times the radius of the base.

43 Find the shortest distance from the point ($x' = 1$, $y' = 2$) to the line $3y = 4x + 12$. Ans. 2.

44. Find the shortest distance from the point ($x' = 2$, $y' = 1$) to the parabola $y^2 = 4x$. $\sqrt{2}$

45. A square sheet of tin has a square cut out at each corner; find the side of the square cut out when the remainder of the sheet will form an open-top box of maximum capacity.

A side $= \frac{1}{6}$ the side of the sheet of tin.

46. A man is at one corner of a square field whose sides are each 780 yards and wishes to go to the opposite corner in the least possible time; (1) how far along the side must he go before turning across the field if he can travel along the side and through the field at the rates, respectively, of 65 and 25 yards per minute? (2) In what time will he reach the opposite corner? (1) 455 yards; (2) 40 min. 48 sec.

47. Find the altitude of the least isosceles triangle circumscribed about an ellipse whose semi-axes are a and b, the base of the triangle being parallel to the major axis. $3b$.

48. A steamer whose speed is 8 knots per hour and course due north sights another steamer directly ahead, whose speed is 10 knots and whose course is due west. What must be the course of the first steamer to cross the track of the second at the least possible distance from her? N. ($\cos^{-1}\frac{3}{5}$) W.

49 If the statue of Washington on the cupola of the Capitol is a feet in height and b feet above the level of an observer's eyes, at what horizontal distance from the centre of the cupola

should the observer stand to obtain the most favorable view of the statue? $\sqrt{b(a+b)}$ feet.

MAXIMA AND MINIMA OF FUNCTIONS OF TWO INDEPENDENT VARIABLES.

147 Definition.—A function, $u = f(x, y)$, of two independent variables has a maximum or minimum value according as

$$f(x+h, y+k) < f(x, y), \quad \text{or} \quad f(x+h, y+k) > f(x, y),$$

for all small values of h and k, positive or negative.

148. Conditions for maxima and minima.—In the function $u = f(x, y)$ if we suppose x and y to vary simultaneously, it is obvious from Art. 139, that the maximum or minimum values of u will occur at the points where the total differential of u, $[du]$, is equal to zero. That is, when

$$[du] = \frac{du}{dx}dx + \frac{du}{dy}dy = 0. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

As $dx\ (= h)$ and $dy\ (= k)$ are independent of each other, each term of (1) must be equal to zero. Hence

$$\frac{du}{dx} = 0, \quad \text{and} \quad \frac{du}{dy} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

These equations express the first conditions essential to the existence of either a maximum or a minimum.

Again, as u passes through a maximum or minimum value, $[du]$ changes from $+$ to $-$, or $-$ to $+$, respectively; therefore, in the former case $[du]$ is decreasing, hence $[d^2u]$ is $-$, and in the latter $[du]$ is increasing, hence $[d^2u]$ is $+$. But the signs of

$$[d^2u] = \frac{d^2u}{dx^2}dx^2 + \frac{2d^2u}{dx\,dy}dx\,dy + \frac{d^2u}{dy^2}dy^2. \quad . \quad . \quad (3)$$

must evidently be independent of the signs of dx and dy, however large or small these differentials may be supposed to be. This can be the case only when

$$\left(\frac{d^2u}{dx^2}\right)\left(\frac{d^2u}{dy^2}\right) > \left(\frac{d^2u}{dx\,dy}\right)^2. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For, making $A = \frac{d^2u}{dx^2}$, $B = \frac{d^2u}{dx\,dy}$. $C = \frac{d^2u}{dy^2}$, we have

$$Ah^2 + 2Bhk + Ck^2 = \frac{(Ah + Bk)^2 + (AC - B^2)k^2}{A}. \quad . \quad (5)$$

In order that (5) may preserve the same sign for all values of h and k, it is necessary that $AC - B^2$ should be positive; for if negative, the numerator of (5) will be positive when $k = 0$, and negative when $Ah + Bk = 0$. Hence we have as an additional condition for a maximum or a minimum, $AC > B^2$, or (4).

With this condition, the sign of (5) depends on that of the denominator A. Hence for a maximum we must have

$$A \text{ or } \frac{d^2u}{dx^2} < 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and for a minimum

$$A \text{ or } \frac{d^2u}{dx^2} > 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

It should be noticed that $AC > B^2$ requires that A and C should have the same sign.

The exceptional cases, where $B^2 = AC$, or where $A = 0$, $B = 0$, $C = 0$, require further investigation, but we shall not consider them in this book.

The conditions for a maximum or minimum value of $u = f(x, y)$ are then, viz.:

For either a maximum or a minimum,

$$\frac{du}{dx} = 0, \quad \text{and} \quad \frac{du}{dy} = 0; \quad . \quad . \quad . \quad . \quad . \quad (8)$$

also
$$\frac{d^2u}{dx^2}\frac{d^2u}{dy^2} > \left(\frac{d^2u}{dx\,dy}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

For a maximum,
$$\frac{d^2u}{dx^2} < 0, \quad \frac{d^2u}{dy^2} < 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

For a minimum,
$$\frac{d^2u}{dx^2} > 0, \quad \frac{d^2u}{dy^2} > 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

EXAMPLES.

1. Find the minimum value of $u = x^3 + y^3 - 3axy$.

Here $$\frac{du}{dx} = 3x^2 - 3ay; \quad \frac{du}{dy} = 3y^2 - 3ax;$$

also $$\frac{d^2u}{dx^2} = 6x, \quad \frac{d^2u}{dy^2} = 6y, \quad \frac{d^2u}{dx\,dy} = -3a.$$

Applying (8), we have
$$x^2 - ay = 0, \quad \text{and} \quad y^2 - ax = 0;$$
whence $$x = 0,\ y = 0; \quad \text{and} \quad x = a,\ y = a.$$

The values $x = 0$, $y = 0$, give
$$\frac{d^2u}{dx^2} = 0, \quad \frac{d^2u}{dy^2} = 0, \quad \frac{d^2u}{dx\,dy} = -3a,$$
which do not satisfy (9). Hence they do not give a maximum or a minimum.

The values $x = a$, $y = a$, give
$$\frac{d^2u}{dx^2} = 6a, \quad \frac{d^2u}{dy^2} = 6a, \quad \frac{d^2u}{dx\,dy} = -3a,$$
which satisfy both (9) and (11).

Hence they give a minimum value of u, which is $-a^3$.

2. Find the minimum value of
$$x^2 + xy + y^2 - ax - by. \qquad \tfrac{1}{3}(ab - a^2 - b^2).$$

3. Find the maximum value of
$$(a - x)(a - y)(x + y - a). \qquad \frac{a^3}{27}.$$

4. Required the triangle of maximum area that can be inscribed in a given circle. The triangle is equilateral.

5. Divide a into three parts, x, y, $a - x - y$, such that their continued product, $xy(a - x - y)$, may be the greatest possible.
$$x = y = a - x - y = \frac{a}{3}.$$

6. Divide 45 into three parts, x, y, $45 - x - y$, such that $x^2y^3(45 - x - y)^4$ may be a maximum.

$x = 10,\ y = 15,\ 45 - x - y = 20.$

7. Find the least possible surface of a rectangular parallelopiped whose volume is a^3. $6a^2$.

8. Find the dimensions of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

$\frac{2}{3}a\sqrt{3},\ \frac{2}{3}b\sqrt{3},\ \frac{2}{3}c\sqrt{3}.$

CHAPTER VII.

APPLICATIONS OF THE DIFFERENTIAL CALCULUS TO PLANE CURVES.

TANGENTS, NORMALS, AND ASYMPTOTES.

149. Equations of the Tangent and Normal. In Fig. 23 let $P(x_1, y_1)$ be the point of tangency of the tangent TP; then the equation of TP is $y - y_1 = m(x - x_1)$, where m is the tangent of the angle BTP. But, Art. 26, $\tan BTP = \frac{dy_1}{dx_1}$; therefore the equation of the tangent PT is

$$y - y_1 = \frac{dy_1}{dx_1}(x - x_1), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A)}$$

where $\frac{dy_1}{dx_1}$ is the value of $\frac{dy}{dx}$ with respect to the curve AP at the point (x_1, y_1).

Since the normal PN is perpendicular to the tangent or curve at P, its equation may be obtained from (A) by substituting $-\frac{dx_1}{dy_1}$ for $\frac{dy_1}{dx_1}$, which gives

$$y - y_1 = -\frac{dx_1}{dy_1}(x - x_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(B)}$$

EXAMPLES.

1. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$.

Here $$\frac{dy}{dx} = \frac{2a}{y}; \quad \therefore \frac{dy_1}{dx_1} = \frac{2a}{y_1}.$$

Substituting this value of $\frac{dy_1}{dx_1}$ in (A) and (B), we have

$$y - y_1 = \frac{2a}{y_1}(x - x_1), \text{ tangent}; \quad . \quad . \quad . \quad . \quad (1)$$

$$y - y_1 = -\frac{y_1}{2a}(x - x_1), \text{ normal}. \quad . \quad . \quad . \quad (2)$$

2. Find the equations of the tangent and normal to the parabola $y^2 = 18x$ at the point $x_1 = 2$.

Here $4a = 18$ and $y_1^2 = 18x_1$; $\therefore 2a = 9$ and $y_1 = 6$. Substituting in (1) and (2), and reducing, we have

$$2y = 3x + 6, \text{ tangent}, \quad \text{and} \quad 3y = -2x + 22, \text{ normal}.$$

Find the equations of the tangent and normal to the following curves:

3. The circle, $y^2 + x^2 = R^2$.

$$(1)\ yy_1 + xx_1 = R^2; \quad (2)\ yx_1 - xy_1 = 0.$$

4. The ellipse, $a^2y^2 + b^2x^2 = a^2b^2$.

$$(1)\ a^2yy_1 + b^2xx_1 = a^2b^2; \quad (2)\ y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1).$$

5. The cissoid, $y^2(2a - x) = x^3$.

$$(1)\ \text{tangent}, y - y_1 = \pm\frac{(3a - x_1)\sqrt{x_1}}{(2a - x_1)^{\frac{3}{2}}}(x - x_1).$$

6. Find the equation of the normal to $y^2 = 6x - 5$ at $y_1 = 5$.

$$y = -\tfrac{5}{3}x + \tfrac{40}{3}.$$

7. Find the equation of the tangent at $y_1 = 4$ to the cycloid

$$x = 10 \operatorname{vers}^{-1}\frac{y}{10} - \sqrt{20y - y^2}.$$

$$y = 2x + 20(1 - \operatorname{vers}^{-1}\tfrac{2}{5}).$$

150. Length of Tangent, Normal, Subtangent, and Subnormal. Let PT represent the tangent at the point $P(x_1, y_1)$, PN the normal; then $y_1 = PB$, BT is the subtangent, and BN is the subnormal.

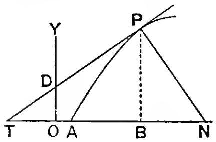

FIG. 23.

Let $\phi =$ the angle BTP, then

(1) $$BT = BP \cot\phi = y_1\frac{dx_1}{dy_1};$$

that is, $$\text{subtangent} = y_1\frac{dx_1}{dy_1}. \quad \ldots \ldots \quad \text{(C)}$$

(2) $$BN = BP \tan BPN = BP \tan\phi;$$

that is, $$\text{subnormal} = y_1\frac{dy_1}{dx_1}. \quad \ldots \ldots \quad \text{(D)}$$

(3) $$TP = (BP \div \sin\phi) = BP\frac{ds_1}{dy_1}; \qquad \text{(Art. 49)}$$

that is, $$\text{tangent} = y_1\frac{ds_1}{dy_1}. \quad \ldots \ldots \quad \text{(E)}$$

(4) $$PN = (BP \div \cos\phi) = BP\frac{ds_1}{dx_1};$$

that is, $$\text{normal} = y_1\frac{ds_1}{dx_1}. \quad \ldots \ldots \quad \text{(F)}$$

In formulas (E) and (F), $ds_1 = \sqrt{dx_1^2 + dy_1^2}$, Art. 33.

If the subtangent be estimated from the point T, and the subnormal from B, each will be positive or negative according as it extends to the right or left.

EXAMPLES.

1. Find the values of the subtangent and subnormal of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$.

Here $\dfrac{dy_1}{dx_1} = -\dfrac{b^2x_1}{a^2y_1}$; substituting in (C) and (D) we have

$$\text{Subt.} = -\frac{a^2y_1^2}{b^2x_1} = \frac{x_1^2 - a^2}{x_1}; \quad \text{subn.} = -\frac{b^2x_1}{a^2}.$$

2. Find the values of the subtangent and subnormal of the ellipse $9y^2 + 4x^2 = 36$, at $x_1 = 1$.

Here $a = 3$, $b = 2$, $x_1 = 1$, which substituted in the preceding answers give subt. $= -8$, subn. $= -\frac{4}{9}$.

Find the values of the subtangents and subnormals of the following:

3. $y^3 = ax$. Subt. $= 3x_1$; subn. $= \dfrac{a}{3y_1}$.

4. Parabola, $y^2 = 4ax$. Subt. $= 2x_1$; subn. $= 2a$.

5. $y = a^x$. Subt. $= \dfrac{1}{\log a}$; subn. $= a^{2x_1} \log a$.

6. Find the length of the tangent of the tractrix,

$$x = a \log\left(\frac{a + \sqrt{a^2 - y^2}}{y}\right) - (a^2 - y^2)^{\frac{1}{2}}. \qquad \text{Tang.} = a.$$

7. Find the lengths of the normal and subnormal of the cycloid $x = r \operatorname{vers}^{-1}\dfrac{y}{r} - \sqrt{2ry - y^2}$.

Norm. $= \sqrt{(2ry)}$; subn. $= \sqrt{(2ry - y^2)}$.

151. Lengths of Tangent, Normal, Subtangent, and Subnormal in Polar Co-ordinates. Let AP $(= s)$ be a curve, O the pole, OP $(= r)$ the radius vector, PT a tangent at P. Let $\theta = XOP$.

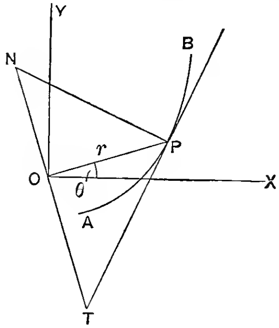

FIG. 24.

Draw OT perpendicular to OP and prolong it to meet the normal NP at N; then PT is the polar tangent, PN the polar normal, OT the polar subtangent, and ON the polar subnormal. Evidently,

$$ONP = OPT = \psi. \qquad \text{(Art. 97)}$$

(1) $$OT = OP \tan OPT = r\left(\frac{rd\theta}{dr}\right); \qquad \text{(Art. 98)}$$

that is, $$\text{the polar subtangent} = \frac{r^2 d\theta}{dr}. \quad \ldots\ldots \quad \text{(G)}$$

(2) $$ON = OP \cot ONP = r\left(\frac{dr}{rd\theta}\right);$$

that is, $$\text{the polar subnormal} = \frac{dr}{d\theta}. \quad \ldots\ldots \quad \text{(H)}$$

(3) $$TP = (OP \div \cos OPT) = r \div \frac{dr}{ds};$$

that is, $$\text{the polar tangent} = \frac{rds}{dr}. \quad \ldots\ldots \quad \text{(I)}$$

(4) $$PN = (OP \div \sin ONP) = r \div \frac{rd\theta}{ds};$$

that is, $$\text{the polar normal} = \frac{ds}{d\theta}. \quad \ldots\ldots \quad \text{(J)}$$

In formulas (I) and (J) $ds = \sqrt{dr^2 + r^2 d\theta^2}$, Art. 97.

EXAMPLES.

Find the tangent, normal, subtangent, and subnormal of the following polar curves:

1. The spiral of Archimedes, $r = a\theta$.

$$\frac{d\theta}{dr} = \frac{1}{a};$$

$$\text{From (G), subt.} = \frac{r^2}{a}; \quad \text{from (H), subn.} = a;$$

$$\text{from (I), tang.} = \frac{r}{a}\sqrt{a^2 + r^2};$$

$$\text{from (J), norm.} = \sqrt{a^2 + r^2}.$$

2 The logarithmic spiral, $r = a^\theta$.

$$\frac{dr}{d\theta} = a^\theta \log a.$$

Substituting in (G), (H), (I), (J), we find

$$\text{subt.} = \frac{r}{\log a} = mr; \quad \text{subn.} = \frac{r}{m};$$

$$\text{tang.} = r\sqrt{1 + m^2}; \quad \text{norm.} = r\sqrt{1 + \log^2 a}.$$

Find the subtangent and subnormal of the following:

3. The hyperbolic spiral, $r\theta = a$.

$$\text{Subt.} = -a; \text{ subn.} = -\frac{r^2}{a}.$$

4. The Lemniscate of Bernouilli, $r^2 = a^2 \cos 2\theta$.

$$\text{Subt.} = \frac{-r^3}{a^2 \sin 2\theta}; \quad \text{subn.} = -\frac{a^2}{r} \sin 2\theta.$$

152. An **Asymptote** to a curve is a tangent which passes within a finite distance of the origin and touches the curve at an infinite distance. A curve which has no infinite branch can have no real asymptote.

In Fig. 23, let x_0 and y_0 represent the intercepts OT and OD, respectively; then, in (A), Art. 149, by making (1) $y = 0$ and (2) $x = 0$, we find

$$(1)\ x_0 = x_1 - y_1\frac{dx_1}{dy_1} = OT; \quad . \quad . \quad . \quad . \quad \text{(K)}$$

$$(2)\ y_0 = y_1 - x_1\frac{dy_1}{dx_1} = OD. \quad . \quad . \quad . \quad . \quad . \quad \text{(L)}$$

Now, if the curve AP is of such a character that x_0 or y_0, or both, remain finite when x_1 or y_1, or both, become infinite (see Art. 154), the tangent TP will be an asymptote to the curve.

EXAMPLES.

1. Examine $y^3 = 6x^2 + x^3$ for asymptotes.

Since $\dfrac{dy}{dx} = \dfrac{4x + x^2}{y^2}$, $\dfrac{dy_1}{dx_1} = \dfrac{4x_1 + x_1^2}{y_1^2}$, which substituted in (K) and (L) give

$$(1)\ x_0 = x_1 - \frac{y_1^3}{4x_1 + x_1^2} = -\frac{2}{\dfrac{4}{x_1} + 1},$$

which $= -2$ when $x_1 = \infty$.

$$(2)\ y_0 = y_1 - \frac{4x_1^2 + x_1^3}{y_1^2} = \frac{2}{\left(\dfrac{6}{x_1} + 1\right)^{\frac{2}{3}}},$$

which $= 2$ when $x_1 = \infty$.

Therefore the straight line whose x and y intercepts are -2 and $+2$, respectively, is an asymptote to the curve.

Since the asymptote passes through the points $(-2, 0)$ and $(0, 2)$, its equation is $y = x + 2$.

153. General Equation of the Asymptote. Since the asymptote passes through the points $(x_0, 0)$ and $(0, y_0)$, its equation is

$$y = \frac{dy_1}{dx_1}(x - x_0), \quad . \; . \; . \quad (M), \quad \text{or} \quad y = \frac{dy_1}{dx_1}x + y_0. \quad . \; . \; . \quad (N)$$

This equation enables us to determine whether or not any given curve has an asymptote, and, if it has, to find its equation. Let us denote the values which $\frac{dy_1}{dx_1}$ and y_0 assume when $x_1 = \infty$ by m_1 and b_1, respectively; then we have

$$y = m_1 x + b_1. \quad . \; . \; . \; . \; . \; . \; . \; . \quad (P)$$

154. When the terms of the equation $f(x, y) = 0$ are of different degrees, to find the relation of y to x when they are infinite, we may omit all the terms except the group which are of the highest degree with respect to x and y.

Thus, when x is infinite, the equation $ay^2 - bx^2 + cy + dx = e$ gives $ay^2 - bx^2 = 0$, or $y = \pm\sqrt{\frac{b}{a}}x$.

A curve like $a^2y^2 + b^2x^2 = a^2b^2$, or $y^4 = x^2(a^2 - x^2)$, etc., which has no infinite branch or branches, has no real asymptote; this is indicated by the fact that when x is infinite, y, as determined above, will be imaginary.

2. Find the asymptote of the hyperbola $a^2y^2 - b^2x^2 = -a^2b^2$.

When $x = \infty$, $\quad y = \pm\frac{b}{a}x, \quad$ or $\quad y_1 = \pm\frac{b}{a}x_1$.

$$\frac{dy_1}{dx_1} = \frac{b^2x_1}{a^2y_1}; \quad y_0 = y_1 - \frac{b^2x_1^2}{a^2y_1} = -\frac{b^2}{y_1}.$$

$\therefore m_1 = \pm \frac{b}{a}$ and $b_1 = 0$, which substituted in (P) gives

$$y = \pm \frac{b}{a}x, \text{ Ans.}$$

3. Find the asymptote of the parabola $y^2 = 4ax$.

When $x = \infty$, $\quad y = \pm 2\sqrt{ax}, \quad$ or $\quad y_1 = \pm 2\sqrt{ax_1}$.

$$\frac{dy_1}{dx_1} = \frac{2a}{y_1}; \quad y_0 = y_1 - \frac{2ax_1}{y_1} = \sqrt{ax_1}, \text{ which} = \infty \text{ when } x_1 = \infty.$$

Therefore the parabola has no asymptote.

4. Find the asymptote of $y^3 = ax^2 + x^3$.

When $x = \infty$, we have $y^3 = x^3$; $\therefore y = x$ or $y_1 = x_1$.

$$\frac{dy_1}{dx_1} = \frac{2ax_1 + 3x_1^2}{3y_1^2}, \quad \text{and} \quad y_0 = \frac{ax_1^2}{3y_1^2};$$

hence $m_1 = 1$ and $b_1 = \frac{a}{3}$, and the asymptote is $y = x + \frac{a}{3}$.

155. Asymptotes Determined by Inspection. When an asymptote is perpendicular to the axis of x or y, it can often be determined by inspection. In the first case $m_1 = \infty$, or $\frac{dx_1}{dy_1} = 0$, which, substituted in (M), gives $x - x_1 = 0$, since, in this case, $x_0 = x_1$; that is, if y_1 is *infinite* when x_1 is finite, $x - x_1 = 0$ is the equation of the asymptote.

Thus, in the cissoid, $\quad y^2 = \dfrac{x^3}{2a - x}$,

$y = \infty$ when $x = 2a$; hence the line $x - 2a = 0$, which is paral-

lel to the axis of y and at a distance $2a$ from it, is an asymptote to the curve.

Again, in $xy = a$ or $y = \frac{a}{x}$, when $x = 0$, $y = \infty$; therefore $x = 0$, or the axis of y, is an asymptote to the curve.

Similarly, in $y = a^x$, when $x = -\infty$, $y = 0$; hence $y = 0$, or the axis of x, is an asymptote to the logarithmic curve.

5. Find the asymptotes of $xy - ay - bx = 0$.

$$(1)\ x - a = 0; \quad (2)\ y - b = 0.$$

6. Find the asymptote of $y^3 = ax^2 - x^3$.

$$y = -x + \frac{a}{3}.$$

7. Find the asymptotes of $y = c + \dfrac{a^3}{(x - b)^2}$.

$$y = c \text{ and } x = b.$$

8. Find the asymptotes of $y^2(x^2 + 1) = x^2(x^2 - 1)$.

$$y = \pm x.$$

9. Find the asymptotes of $y^2(x - a) = x^3 + ax^2$.

$$x = a \text{ and } y = \pm (x + a).$$

CURVATURE.

156. A point moving along an arc of a curve changes its direction continuously, and the *total change of direction* is called the **Total Curvature** of the arc.

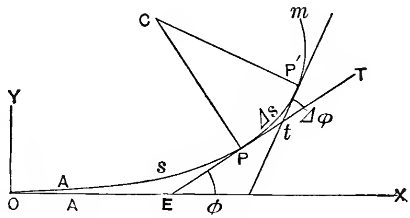

FIG. 25.

The angle TtP', Fig. 25, through which the tangent PT rotates as the point of tangency P moves from P to P', being

the total change of direction of the point P, is the total curvature of the arc PP'.

157. Uniform Curvature. The curvature is uniform when, as the point of tangency moves over equal arcs, the tangent turns through equal angles; that is, when the distance described by the point varies as its direction.

Let APm be the curve, $AP = s$, $PP' = \Delta s$, $XEP = \phi$, Art. 49; then $TtP' = \Delta\phi$. Let PC and $P'C$ be normals meeting at C.

Supposing $\Delta s \propto \Delta\phi$, we have (Art. 12)

$$\Delta s = m\Delta\phi, \quad \text{or} \quad \frac{1}{m} = \frac{\Delta\phi}{\Delta s}.$$

(1) Let us consider the meaning of $\dfrac{\Delta\phi}{\Delta s}$. If the distance Δs gives a total curvature of $\Delta\phi$, since $\Delta s \propto \Delta\phi$, a distance of 1 will give a curvature of $\dfrac{\Delta\phi}{\Delta s}$. That is, $\dfrac{\Delta\phi}{\Delta s}$ is the curvature per distance of unity, or the rate of change of the direction of a curve with respect to that of its length, for which reason it is called the curvature of the curve.

(2) Let us determine the value of m. The circle is the only curve of uniform curvature. Hence, supposing $\Delta s \propto \Delta\phi$, PP' is the arc of a circle whose radius (say R) is CP. The angle $PCP' = TtP' = \Delta\phi$; but arc $PP' = CP \times$ angle PCP'; that is, $\Delta s = R\Delta\phi$; hence $m = R$, and we have

$$\frac{\Delta\phi}{\Delta s} = \frac{1}{R}.$$

Cor. I. The curvature of any circle is equal to the reciprocal of its radius; and the curvatures of any two circles are inversely proportional to their radii.

Cor. II. If $R = 1$, $\dfrac{\Delta\phi}{\Delta s} = 1$; that is, the unit of curvature is the curvature of a circle whose radius is unity.

158. Variable Curvature. When the curvature is variable, we define the curvature at any point P of the curve as the value which $\frac{\Delta\phi}{\Delta s}$ would have were the curvature there to become uniform. Hence the curvature at P is the value of $\frac{d\phi}{ds}$ at that point.

159. Radius of Curvature. A circle tangent to a curve at any point, and having the same curvature as that of the curve at that point, is called the circle of curvature; its radius, the radius of curvature; and its centre, the centre of curvature.

The curvature of this circle being that of the given curve, is equal to $\frac{d\phi}{ds}$; therefore the radius of curvature of APm at P, Fig. 25, is

$$R = \frac{ds}{d\phi}.$$

COR. I. *To express R in terms of the differentials of x and y.*

$$\tan\phi = \frac{dy}{dx}, \qquad \therefore \phi = \tan^{-1}\frac{dy}{dx};$$

hence we have

$$d\phi = \frac{d^2y\,dx}{dx^2 + dy^2}; \quad \text{also,} \quad ds = (dx^2 + dy^2)^{\frac{1}{2}}.$$

$$\therefore R^* = \frac{ds}{d\phi} = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx\,d^2y}, \quad \text{or} \quad \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}. \quad . \quad (1)$$

---

* R will be positive or negative according as the curve is convex or concave (Art. 173), but its sign is often neglected.

EXAMPLES.

1. Find the radius of curvature of the parabola $y^2 = 4ax$.

Here $\dfrac{dy}{dx} = \dfrac{2a}{y}$, and $\dfrac{d^2y}{dx^2} = -\dfrac{4a^2}{y^3}$.

Substituting in (1), we have

$$R = \frac{(y^2 + 4a^2)^{\frac{3}{2}}}{4a^2}.$$

At the vertex, where $y = 0$, we have $R = 2a$, which is evidently the minimum radius of curvature.

2. Find the radius of curvature of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$.

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}, \quad \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3};$$

$$\therefore \quad R = \frac{a^2y^3}{b^4}\left(\frac{a^4y^2 + b^4x^2}{a^4y^2}\right)^{\frac{3}{2}} = \frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^4b^4}.$$

At the vertex $x = a$, $y = 0$, $R = \dfrac{b^2}{a}$, and at the vertex $x = 0$, $y = b$, $R = \dfrac{a^2}{b}$, which are respectively the minimum and maximum radii of curvature.

3. Find the radius of curvature of the cycloid

$$x = r \operatorname{vers}^{-1}\frac{y}{r} - \sqrt{2ry - y^2}.$$

Here $\dfrac{dx}{dy} = \dfrac{y}{\sqrt{2ry - y^2}}$; $\therefore\ 1 + \dfrac{dy^2}{dx^2} = \dfrac{2r}{y}$;

$$\frac{d^2y}{dx^2} = -\frac{r}{y^2}; \quad \therefore R = 2\sqrt{2ry},$$

which equals twice the normal.

4. Find the radius of curvature of the logarithmic curve $y = a^x$.

$$R = \frac{(m^2 + y^2)^{\frac{3}{2}}}{my}.$$

5. Find the point on the parabola $y^2 = 8x$ at which the radius of curvature is $7\frac{13}{16}$. $y = 3,\ x = 1\frac{1}{8}$.

6. Find the radius of curvature of $y = x^4 - 4x^3 - 18x^2$ at the origin.

$$R = \frac{1}{36}.$$

7. Find the curvature of the equilateral hyperbola $xy = 12$ at the point where $x = 3$.

$$\frac{1}{R} = \frac{24}{125}.$$

8. Find the radius of curvature of the catenary

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right).$$

$$R = \frac{y^2}{a}.$$

9. Find the radius of curvature of the hypocycloid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}. \qquad R = 3(axy)^{\frac{1}{3}}.$$

160. The radius of curvature in polar co-ordinates can be found by transforming the value of R given in the answer to Ex. 7, Art. 112, to polar co-ordinates. We thus obtain

$$R = \frac{\left(r^2 + \frac{dr^2}{d\theta^2}\right)^{\frac{3}{2}}}{r^2 + 2\frac{dr^2}{d\theta^2} - r\frac{d^2r}{d\theta^2}}. \quad \ldots \ldots \quad (2)$$

EXAMPLES.

1. Find the radius of curvature of the spiral of Archimedes, $r = a\theta$.

Here $$\frac{dr}{d\theta} = a, \quad \frac{d^2r}{d\theta^2} = 0;$$

substituting in (2), we have

$$R = \frac{(r^2 + a^2)^{\frac{3}{2}}}{r^2 + 2a^2} = \frac{a(1 + \theta^2)^{\frac{3}{2}}}{2 + \theta^2}.$$

Find the radius of curvature of the following:

2. The logarithmic spiral $r = a^\theta$. $\qquad R = r\sqrt{1 + (\log a)^2}$

3. The cardioid $r = a(1 - \cos\theta)$. $\qquad R = \frac{2}{3}\sqrt{2ar}.$

4. The lemniscate $r^2 = a^2\cos 2\theta$. $\qquad R = \frac{a^2}{3r}.$

CONTACT OF DIFFERENT ORDERS.

161. Let $y = f(x)$ and $y = \phi(x)$ be any two curves referred to the same axes. Let the curves intersect at the point P, whose abscissa is a, then $f(a) = \phi(a)$. If $f(a) = \phi(a)$, and $f'(a) = \phi'(a)$, the curves are tangent at P, and are said to have a contact of the first order. If $f(a) = \phi(a), f'(a) = \phi'(a)$, and $f''(a) = \phi''(a)$, the curves have the same curvature at P, and their contact is of the second order. If, in addition, $f'''(a) = \phi'''(a)$, their contact is of the third order; and so on. Thus, contact of the nth order imposes $n + 1$ conditions.

162. *Two curves cross or do not cross at their point of contact, according as their order of contact is even or odd.*

Let $x = a$ be the abscissa of the point of contact of the

curves $y = f(x)$ and $y = \phi(x)$, then $f(a) = \phi(a)$. Let h be a small increment of x. By Taylor's formula, we have

$$f(a+h) = f(a) + f'(a)h + f''(a)\frac{h^2}{\underline{|2}} + f'''(a)\frac{h^3}{\underline{|3}} + ; \quad (1)$$

$$\phi(a+h) = \phi(a) + \phi'(a)h + \phi''(a)\frac{h^2}{\underline{|2}} + \phi'''(a)\frac{h^3}{\underline{|3}} + . \quad (2)$$

Subtracting (2) from (1), we obtain

$$f(a+h) - \phi(a+h) = h[f'(a) - \phi'(a)] + \frac{h^2}{\underline{|2}}[f''(a) - \phi''(a)]$$

$$+ \frac{h^3}{\underline{|3}}[f'''(a) - \phi'''(a)] + \frac{h^4}{\underline{|4}}[f^{\text{iv}}(a) - \phi^{\text{iv}}(a)] + . \quad (3)$$

(*a*) If $f(x) - \phi(x)$ changes sign as x increases from $a - h$ to $a + h$, the two curves evidently cross at a; if not, the curves touch each other, but do not cross.

(*b*) If the contact is of an odd order, the first term of the second member of (3), which does not vanish, contains an even power of h; hence the sign of the second member, and therefore the first, undergoes no change as x passes from $a - h$ to $a + h$, and the curves do not cross.

(*c*) If the contact is of an even order, the first term of the second member of (3), which does not vanish, contains an odd power of h; hence, in this case, $f(x) - \phi(x)$ changes sign as x passes from $a - h$ to $a + h$, and therefore the curves cross.

Cor. I. At a point of maximum or minimum curvature, the circle of curvature has contact of the third order with the curve, for it does not cut the curve at such a point.

Cor. II. If two curves are tangent to, and cross each other at, a certain point, they have contact of at least the second order.

EXAMPLES.

1. Find the order of contact of the two curves

$$y = x^3 - 3x^2 + 7 \quad \text{and} \quad y + 3x = 8.$$

By combining the two equations we find that ($x = 1$, $y = 5$) is a point of contact.

Making $f(x) = x^3 - 3x^2 + 7$ and $\phi(x) = 8 - 3x$, we have

$$f'(x) = 3x^2 - 6x, \quad \phi'(x) = -3; \quad \therefore \quad f'(1) = \phi'(1) = -3;$$

$$f''(x) = 6x - 6, \quad \phi''(x) = 0; \quad \therefore f''(1) = \phi''(1) = 0;$$

$$f'''(x) = 6, \quad \phi'''(x) = 0; \quad \therefore f'''(1) > \phi'''(1).$$

Hence the contact is of the second order.

2. Find the order of contact of the parabola $y^2 = 4x$ and the line $3y = x + 9$. First order.

3. Find the order of contact of the curves

$$y = 3x - x^2 \quad \text{and} \quad xy = 3x - 1. \qquad \text{Second order.}$$

4. Find the order of contact of

$$y = \log(x - 1) \quad \text{and} \quad x^2 - 6x + 2y + 8 = 0,$$

at the point (2, 0). Second order.

5. Find the order of contact of the parabola $y^2 = 4x + 4$ and the circle $y^2 + x^2 = 2x + 3$. Third order.

163. Osculating Curves. The curve of a given species that has the highest order of contact possible with a given curve at any point is called the osculating curve of that species.

A curve may be made to fulfil as many independent conditions as there are arbitrary constants in its equation, and no more. Therefore, in order that $y = f(x)$ may have contact of

the nth order with a given curve at a given point, the equation must involve $n+1$ arbitrary constants.

Hence, as $y = ax + b$ has two constants, the osculating straight line has contact of the first order.

As $(x-a)^2 + (y-b)^2 = r^2$ has three constants, the osculating circle has, in general, contact of the second order.

164. To find the osculating straight line at any point (x', y') of a given curve $y = f(x)$.

The equation of a line is

$$y = ax + b. \quad \dots \dots \dots \quad (1)$$

Since the line and curve pass through (x', y'), we have

$$y' = ax' + b = f(x'). \quad \dots \dots \dots \quad (2)$$

Also, $$\frac{dy'}{dx'} = a \qquad = f'(x'), \quad \dots \dots \dots \quad (3)$$

since $f'(x') = \phi'(x')$.

Solving (2) and (3) for a and b, we have

$$a = \frac{dy'}{dx'}, \quad \text{and} \quad b = y' - \frac{dy'}{dx'}x',$$

which, substituted in (1), gives

$$y - y' = \frac{dy'}{dx'}(x - x').$$

Therefore the osculating straight line is a tangent to the curve, as would be inferred.

165. To find the radius of the osculating circle at any point of a given curve, $y = f(x)$.

The general equation of a circle whose radius is r is

$$(x-a)^2 + (y-b)^2 = r^2. \quad \dots \dots \dots \quad (1)$$

Differentiating twice successively, we have

$$x - a + (y - b)\frac{dy}{dx} = 0, \quad \ldots \ldots \quad (2)$$

$$1 + \frac{dy^2}{dx^2} + (y - b)\frac{d^2y}{dx^2} = 0. \quad \ldots \ldots \quad (3)$$

From (3), $$y - b = -\frac{dx^2 + dy^2}{d^2y}. \quad \ldots \ldots \quad (4)$$

From (2), $$x - a = \frac{(dx^2 + dy^2)dy}{d^2y\,dx}. \quad \ldots \ldots \quad (5)$$

Substituting (4) and (5) in (1), we have

$$r = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx\,d^2y}.$$

By comparing this result with formula 1, Art. 159, it will be seen that the osculating circle is the same as the circle of curvature.

INVOLUTES AND EVOLUTES.

166. An **Involute** may be regarded as a curve traced by a point in a thread as it is unwound from another curve, called the **Evolute.**

Thus, imagine a thread stretched around the curve $A_1P_1m_1$ with one end fastened at m_1; if the thread is unwound by carrying the point at A above and around to the right, that point of the thread will trace the involute APm of which $A_1P_1m_1$ is the evolute.

An evolute may have an unlimited number of involutes, for A may be any point on the curve A_1m_1.

In what follows the chief object is to deduce certain properties of the evolute from its involute, or *vice versa*, and for uniformity the co-ordinates of P (the involute) will be represented by x, y, and those of P_1 (the corresponding point of the

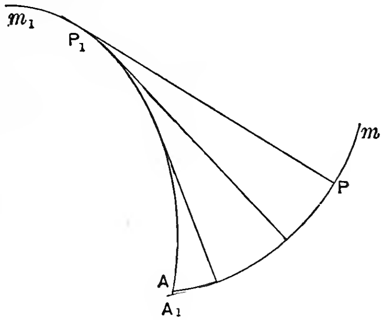

FIG. 26.

evolute) by x_1, y_1; the arc AP by s; the arc AP_1 by s_1; and the angles of direction of AP and AP_1, at P and P_1, by ϕ and ϕ_1, respectively.

167. Elementary Principles. I. $PP_1 =$ the arc $AP_1 = s_1$.

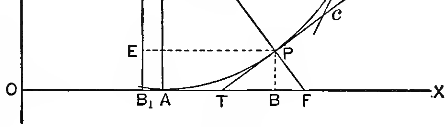

FIG. 27.

II. PP_1 is tangent to AP_1m_1 at P_1, for it has the same direction as the curve at that point.

III. The line PP_1 is a normal to the curve APm at P.

For, draw TP tangent to the curve AP at P.

$$(P_1E)^2 + (EP)^2 = (P_1P)^2, \quad \text{or} \quad (y_1 - y)^2 + (x - x_1)^2 = s_1^2. \quad (1)$$

$$\therefore \quad (y_1 - y)(dy_1 - dy) + (x - x_1)(dx - dx_1) = s_1 ds_1. \quad . \quad (2)$$

Again, $P_1E = P_1P \sin EPP_1$, or $y_1 - y = s_1 \dfrac{dy_1}{ds_1}$; . . (3)

also, $EP = P_1P \cos EPP_1$, or $x - x_1 = - s_1 \dfrac{dx_1}{ds_1}$. . (4)

Substituting in (2) from (3) and (4), and reducing, remembering that $dx_1^2 + dy_1^2 = ds_1^2$, we have $\dfrac{dy_1}{dx_1} = - \dfrac{dx}{dy}$; that is, $\tan \phi_1 = - \cot \phi$; $\therefore \phi_1 = \dfrac{\pi}{2} + \phi$, or $PFX = \dfrac{\pi}{2} + PTX$; hence P_1P is perpendicular to the tangent PT.

Cor. I. Since $\sin \phi_1 = \cos \phi$, $\quad \dfrac{dy_1}{ds_1} = \dfrac{dx}{ds}$; . . . . (5)

also, since $\quad \cos \phi_1 = - \sin \phi$, $\quad \dfrac{dx_1}{ds_1} = - \dfrac{dy}{ds}$. . . . (6)

Cor. II. The point P_1 is the centre of curvature of the curve APm at P.

For, if circles be described from P_1' and P_1 as centres with $P_1'P'$ and P_1P as radii, respectively, the arc PP' will lie within the one circle and without the other, since the straight line $P_1'P'$ is equal to the partly curved line $P_1'P_1P$. Hence the circumference of the circle whose centre is P_1 *crosses* and *touches* the curve APm at P (Art. 162, Cor. II).

Cor. III. Since $P_1P = s_1 = R$, we have (Art. 159)

$$s_1 = \frac{ds}{d\phi} = \frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx d^2 y}. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

COR. IV. From (4) and (6), $x_1 = x - s_1\dfrac{dy}{ds}$, ... (8)

and from (3) and (5), $y_1 = y + s_1\dfrac{dx}{ds}$. . . . . (9)

Substituting for s_1 in (8) and (9), from (7), we have

$$x_1 = x - \frac{\left(1 + \dfrac{dy^2}{dx^2}\right)\dfrac{dy}{dx}}{\dfrac{d^2y}{dx^2}}, \quad \text{and} \quad y_1 = y + \frac{1 + \dfrac{dy^2}{dx^2}}{\dfrac{d^2y}{dx^2}}. \quad . \quad (10)$$

These values of x_1 and y_1 are the values of the co-ordinates of the centre of curvature at the point P.

168. To find the equation of the evolute of any given curve.

By differentiating the equation of the given curve, and substituting the results in (10), x_1 and y_1 may be expressed in terms of x and y. If, between the equations thus obtained and that of the given curve, x and y be eliminated, the resulting equation involving x_1 and y_1 will be the equation of the evolute.

EXAMPLES.

1. Find the equation of the evolute of the parabola $y^2 = 4ax$.

Here $$\frac{dy}{dx} = \frac{2a}{y}; \quad \frac{d^2y}{dx^2} = -\frac{4a^2}{y^3}.$$

Substituting in (10), we have

$$x_1 = x + \frac{y^2 + 4a^2}{y^2}\cdot\frac{2a}{y}\cdot\frac{y^3}{4a^2} = 3x + 2a;$$

$$\therefore \qquad x = \frac{x' - 2a}{3};$$

$$y_1 = y - \frac{y^2 + 4a^2}{y^2} \cdot \frac{y^3}{4a^2} = -\frac{y^3}{4a^2}.$$

$$\therefore \qquad y = -(2a)^{\frac{2}{3}} y_1^{\frac{1}{3}}.$$

These values of x and y substituted in $y^2 = 4ax$ give

$$y_1^2 = \frac{4}{27a}(x_1 - 2a)^3,$$

which is the equation required; hence the evolute is the semi-cubical parabola.

Cor. I. The length of the arc of the evolute s_1 may be found by formula (7), Art. 167.

2. Find the evolute of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$.

$$(ax_1)^{\frac{2}{3}} + (by_1)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

3. Find the co-ordinates of the centre of curvature of the cubical parabola $y^3 = a^2x$.

$$x_1 = \frac{a^4 + 15y^4}{6a^2y}, \quad y_1 = \frac{a^4y - 9y^5}{2a^4}.$$

4. Find the co-ordinates of the centre of curvature of the catenary $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$. $\qquad x_1 = x - \frac{y}{a}\sqrt{y^2 - a^2}, \quad y_1 = 2y.$

5. Find the co-ordinates of the centre of curvature, and the equation of the evolute, of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

$$x_1 = x + 3\sqrt[3]{xy^2}, \quad y_1 = y + 3\sqrt[3]{x^2y}; \quad (x_1 + y_1)^{\frac{2}{3}} + (x_1 - y_1)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$$

6. Find the evolute of the equilateral hyperbola $xy = m^2$.

$$(x_1 + y_1)^{\frac{2}{3}} - (x_1 - y_1)^{\frac{2}{3}} = (4m)^{\frac{2}{3}}.$$

Note.—First prove that

$$x_1 + y_1 = \frac{m}{2}\left(\frac{m}{x} + \frac{x}{m}\right)^3, \quad \text{and} \quad x_1 - y_1 = \frac{m}{2}\left(\frac{m}{x} - \frac{x}{m}\right)^3,$$

and thence derive the equation of the evolute.

ENVELOPES.

169. Let

$$f(x, y, a) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

be the equation of a curve, a being some constant quantity. If we assign different values to a, we will obtain a series of distinct curves, but all belonging to the same system or family of curves. One of the curves of this family can be obtained by increasing a by h, thus converting (1) into

$$f(x, y, a + h) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If h be supposed indefinitely small, the curves (1) and (2) are said to be consecutive.

The points of intersection of the curves (1) and (2) approach definite limiting positions as h approaches 0, and the locus of these limiting positions, as different values are assigned a, is called the **Envelope** of the system $f(x, y, a) = 0$.

The quantity a which remains constant for any one curve of the series, but varies as we pass from one curve to another, is called the variable parameter of the series.

170. *The envelope of a series of curves is tangent to every curve of the series.*

Let A, B, C be any three curves of the series, A and B intersecting at P, and B and C at P'.

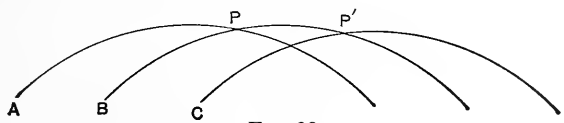

FIG. 28.

As these curves approach coincidence, the limiting positions of P and P' will be two consecutive points of the envelope and of the curve B. Hence the envelope touches B.

As an illustration see example 1 under the next article.

171. To find the equation of the envelope of a given series of curves.

The point of intersection of (1) and (2) will be found by combining the equations. Now, subtracting (1) from (2), we have

$$\frac{f(x, y, a+h) - f(x, y, a)}{h} = 0. \quad . \quad . \quad . \quad . \quad (3)$$

When the curves approach coincidence, h approaches 0, and (3) becomes

$$\frac{d}{da}f(x, y, a) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Thus, equations (1) and (4) determine the intersection of any two consecutive curves. Hence, by eliminating a between (1) and (4), we shall obtain the equation of the locus of these intersections, which is the equation of the envelope.

EXAMPLES.

1. Find the envelope of $y = ax + \frac{m}{a}$, a being the variable parameter.

$y = ax + \frac{m}{a}$ is the equation of a line, as MN, Fig. 29. When a receives an increment h, the line takes a new position, say $M'N'$, which intersects the former line at c. As h approaches 0, c approaches p, a point on the locus (APm) of all similar intersections.

Differentiating with respect to a, x and y being constants, we have

$$0 = x - \frac{m}{a^2}; \quad \text{whence} \quad a = \pm\sqrt{\frac{m}{x}}.$$

$$\therefore \qquad y = \pm[\sqrt{mx} + \sqrt{mx}], \quad \text{or} \quad y^2 = 4mx.$$

which is the equation of a parabola.

Let it be observed that the problem is the same as that of finding the curve of which $y = ax + \frac{m}{a}$ is the tangent.

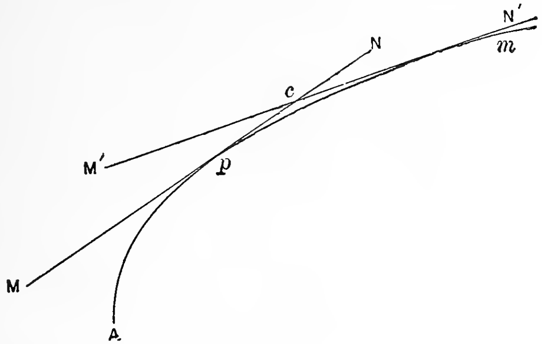

FIG. 29.

2. Find the curve whose tangent is $y = mx + a\sqrt{m^2 + 1}$, m being the variable parameter. $x^2 + y^2 = a^2$, a circle.

3. If a right triangle varies in such a manner that its area is constantly equal to c, find the envelope of the hypothenuse, or the curve to which the hypothenuse is the tangent.

Let $OA = a$, $OB = b$; then the equation of AB is

$$\frac{x}{a} + \frac{y}{b} = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But $ab = 2c$, or $b = \frac{2c}{a}$,

$$\therefore \qquad \frac{x}{a} + \frac{ay}{2c} = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where a is the variable parameter.

Differentiating (2), we get $a = \sqrt{\frac{2cx}{y}}$, which, substituted in (2), gives $xy = \frac{c}{2}$, an equilateral hyperbola.

Solve the preceding problem on the hypothesis that the hypothenuse is constantly equal to c.

$x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$, the hypocycloid.

Since a normal to a curve is tangent to the evolute of the curve, the latter is the envelope of the successive normals, or the locus of their intersections.

4. Find the evolute of the parabola $y^2 = 4ax$, taking the equation to the normal in the form $y = m(x - 2a) - am^3$, m being the variable parameter. $27ay^2 = 4(x - 2a)^3$.

5. Find the evolute of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$, taking the equation of the normal in the form

$$by = ax \tan a' - (a^2 - b^2) \sin a',$$

where the variable parameter a' is the eccentric angle.

$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$. See Ex. 2, Art. 168,

6. Find the curve whose tangent is $y = mx + (a^2m^2 + b^2)^{\frac{1}{2}}$, m being the variable parameter. $a^2y^2 + b^2x^2 = a^2b^2$.

7. Find the envelope of the family of parabolas whose equation is $y^2 = a(x - a)$. $y = \pm \frac{1}{2}x$.

8. Find the locus of the intersections of $x \cos a + y \sin a = p$ with itself as a increases continuously. $x^2 + y^2 = p^2$.

9. Find the envelope of all ellipses having a common area (πc^2), the axes being coincident. $xy = \pm \frac{1}{2}c^2$.

10. Find the evolute of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, the equation of whose normal is $y \cos a' - x \sin a' = a \cos 2a'$, where a' is the angle which the normal makes with the axis of x.

$(x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$. See Ex. 5, Art. 168.

11. Find the equation of the curve, the equation of its tangent being $y = 2mx + m^4$, where m is the variable parameter.

$(\frac{1}{3}y)^3 + (\frac{1}{2}x)^4 = 0$.

TRACING CURVES.

172. The **Rudimentary Method** of tracing a curve is to reduce its equation to the form of $y = f(x)$; that is, solve the

equation $f(x, y) = 0$ for y, assign values to x, find the corresponding values of y, draw a curve through the points thus determined, and it will be approximately the curve required. This process is laborious, and often impossible on account of our inability to solve $f(x, y) = 0$ for y.

The **General Form** of a curve is usually all that is desired, and this can generally be found by determining its singular or characteristic points and properties, and these are embraced chiefly in the position of certain turning-points of the curve, the direction of curvature between these points, and where and how the branches intersect or meet each other. In addition, we may find, by previous methods, where the curve cuts the axes, whether or not it has infinite branches, asymptotes, etc.

173. Direction of Curvature. — The terms **Convex** and **Concave** have their ordinary meaning when applied to the arcs of curves.

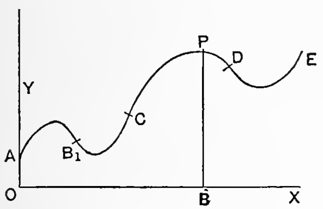

FIG. 30.

Thus, as seen from some point below, the arcs AB_1 and CD are concave, and B_1C and DE convex.

174. A Point of Inflection is the point at which the curve changes from concave to convex, or from convex to concave; as the points B_1, C, D.

PRINCIPLES.—The slope $\left(\frac{dy}{dx}\right)$ of the curve evidently decreases as the point $P(x, y)$ moves from A along the curve to the right until P reaches B_1, and then increases until P reaches C, etc. Therefore (1) when the arc is concave, $\frac{dy}{dx}$ decreases as x increases, hence (Art. 25) its derivative $\frac{d^2y}{dx^2}$ is $-$; (2) when the arc is convex, $\frac{dy}{dx}$ increases as x increases, hence $\frac{d^2y}{dx^2}$ is $+$.

Therefore, I. At any point of the curve $y = f(x)$, the curve is concave or convex according as $\frac{d^2y}{dx^2}$ is negative or positive.

II. The roots of $\frac{d^2y}{dx^2} = 0$ or ∞ which will render $\frac{dy}{dx}$ a maximum or minimum are the abscissas of the points of inflections.

EXAMPLES.

Examine the following curves for concave and convex arcs, and for points of inflection.

1. $y = x^2 - 6x + 7.$ $\qquad \frac{d^2y}{dx^2} = 2.$

Since $\frac{d^2y}{dx^2}$ is $+$, the curve is convex at every point.

2. $y = x^3 - 6x^2 + 17x - 6.$ $\qquad \frac{d^2y}{dx^2} = 6(x - 2).$

The root of $x - 2 = 0$ is 2, the point of inflection; the curve is concave when $x < 2$, convex when $x > 2$.

3. $y = x^4 - 12x^3 + 48x^2 - 50.$

Points of inflection, $x = 2, x = 4$; curve convex when $x < 2$ and > 4, and concave when $2 < x < 4$.

4. $y = \frac{x - 2}{x - 3}.$ $\qquad \frac{d\ y}{dx^2} = \frac{2}{(x - 3)^3}.$

Point of inflection at $x = 3$; convex when $x > 3$, concave when $x < 3$.

5. $y = \log(x - 1).$ $\qquad \frac{d^2y}{dx^2} = -\frac{1}{(x - 1)^2}.$

$\frac{1}{(x - 1)^2} = \infty$ gives $(x - 1)^2 = 0$, which has two equal roots; hence, Art. 140, there is no point of inflection; curve concave.

6. Prove that the curve $y = \dfrac{x^3}{a^2 + x^2}$ has points of inflection at $(0, 0)$, $(a\sqrt{3}, \frac{3}{4}a\sqrt{3})$, $(-a\sqrt{3}, -\frac{3}{4}a\sqrt{3})$.

7. Prove that the witch of Agnesi, $x^2y = 4a^2(2a - y)$, has points of inflection at $(\pm \frac{2}{3}a\sqrt{3}, \frac{3}{2}a)$, and is concave between these points and convex outside of them.

8. Find the points of inflection of $y = \sin 2x + \cos 2x$.

SINGULAR POINTS.

175. The **Singular Points** of a curve are the *turns* and *multiple points*.

A **Turn** in rectangular co-ordinates is a point at which a curve ceases to go (1) up or down, or (2) to the right or left,

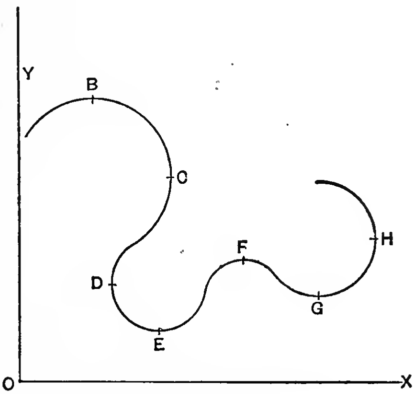

FIG. 31.

and begins to go in the opposite direction. The former, as B, E, F, G, are called **y-turns**, and the latter, as C, D, H, **x-turns.**

The x-turns and y-turns evidently occur at the maximum or minimum values of x and y, respectively.

A **Multiple Point*** is one through which two or more branches of a curve pass, or at which they meet. A multiple point is double when there are only two branches; triple when only three, and so on.

A **Multiple Point of Intersection** is a multiple point at which the branches intersect (Fig. 32, *a*).

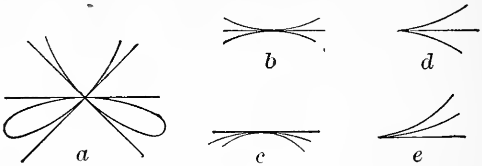

FIG. 32.

An **Osculating Point** is a multiple point through which two branches pass, and at which they are tangent (Fig. 32, *b*, *c*).

A **Cusp** is a multiple point at which two branches terminate and are tangent (Fig. 32, *d*, *e*). A cusp or osculating point is of the first or second species according as the two branches are on opposite sides (Fig. 32, *b*, *d*) or the same side (Fig. 32, *c*, *e*) of their common tangent.

A **Conjugate Point** is one that is entirely isolated from the curve, and yet one whose co-ordinates satisfy the equation of the curve.

For example, in the equation $y = (a + x) \sqrt{x}$, if x is negative y is imaginary, yet the co-ordinates of the point ($x = -a$, $y = 0$) satisfy the equation. Hence $(-a, 0)$ is a conjugate point. A conjugate point is, generally, the intersection or point of meeting of two imaginary branches of the curve, and may, in exceptional cases, also lie on a real branch of the curve.

There are other singular points, such as **Stop Points**, at which a single branch of a curve stops suddenly, and **Shooting Points**, at which two or more branches stop without being tangent to each other. But as these rarely occur, they are omitted in this book.

---

* See Taylor's Calculus.

176. To determine the positions of the singular points of a curve.

Let $u = f(x, y) = 0$ be the equation of the curve, free from radicals. Then (Art. 109)

$$\frac{dy}{dx} = -\frac{\dfrac{du}{dx}}{\dfrac{du}{dy}}.$$

(*a*) For the x-turns, $\dfrac{dy}{dx} = \infty$; $\therefore$ $\dfrac{du}{dy} = 0.$

(*b*) For the y-turns, $\dfrac{dy}{dx} = 0$; $\therefore$ $\dfrac{du}{dx} = 0.$

(*c*) For multiple points, $\dfrac{dy}{dx}$, by definition, has two or more values; hence, since u contains no radicals, $\dfrac{dy}{dx}$ must be of the form $\dfrac{0}{0}$. Therefore

$$\frac{du}{dx} = 0 \quad \text{and} \quad \frac{du}{dy} = 0.$$

Hence, to find the x-turns we have $u = 0$ and $\dfrac{du}{dy} = 0$; to find the y-turns, we have $u = 0$ and $\dfrac{du}{dx} = 0$; and the values of y and x which satisfy all these equations are the co-ordinates of the multiple points.

177. To determine the character of the multiple points of a curve.

From the definitions of the multiple points it follows that:

I. At a multiple point of intersection $\dfrac{dy}{dx}$ has two or more unequal real values.

II. At an osculating point or a cusp $\dfrac{dy}{dx}$ has two equal values.

III. At a conjugate point at least two of the values of $\dfrac{dy}{dx}$ are imaginary.

EXAMPLES.

Find the singular points of the following curves.

1. $$u = x^2 - xy + y^2 - 3 = 0. \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{du}{dy} = -x + 2y = 0; \quad . \quad . \quad (2) \qquad \frac{du}{dx} = 2x - y = 0. \quad . \quad . \quad (3)$$

From (1) and (2) we find (2, 1), (− 2, − 1), the co-ordinates. of the x-turns A and A'.

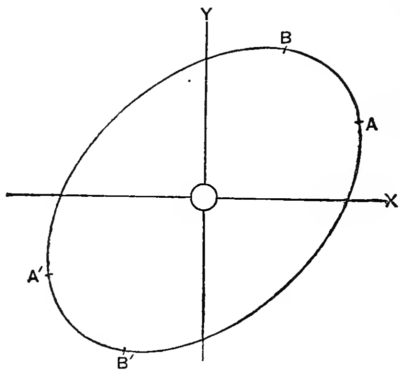

Fig. 33.

From (1) and (3) we find (1, 2), (− 1, − 2), the co-ordinates of the y-turns B and B'.

Since neither pair of these values satisfies (1), (2), (3), the curve has no multiple points.

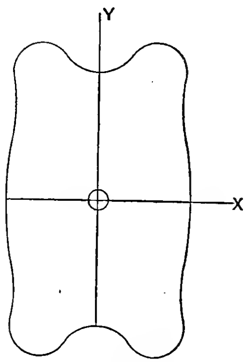

Fig. 34.

2. $$u = 4y^2 - (25 - x^2)(x^2 + 7) = 0.$$

$$\frac{du}{dy} = 8y = 0;$$

$$\frac{du}{dx} = 2x(x^2 + 7) - 2x(25 - x^2) = 0.$$

From these equations we find

(*a*) the x-turns, $(5, 0)$, $(-5, 0)$, $(\pm\sqrt{-7}, 0)$;

(*b*) the y-turns, $(3, \pm 8)$, $(0, \pm \frac{5}{2}\sqrt{7})$, $(-3, \pm 8)$.

The figure (34) is only an approximate representation of the curve.

3. $u = y^2 - 2(2 + x^2)y + \left(\frac{9}{8}x^2\right)^2 = 0.$

$$x\text{-turns, } (\pm 4,\ 18), \quad \text{and} \quad \left(\pm \frac{4}{17}\sqrt{-17},\ \frac{18}{17}\right).$$

$$y\text{-turns, } (0,\ 4),\ (0,\ 0), \quad \text{and} \quad \left(\pm \frac{16}{17}\sqrt{17},\ 19\tfrac{1}{17}\right).$$

4. $\quad u = x^4 + 2ax^2y - ay^3 = 0. \quad . \quad (1)$

$$\frac{du}{dy} = a(2x^2 - 3y^2) \qquad = 0. \quad . \quad (2)$$

$$\frac{du}{dx} = 4x(x^2 + ay) \qquad = 0. \quad . \quad (3)$$

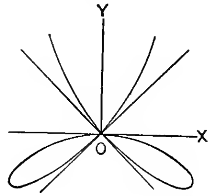

FIG. 35.

$\therefore$ y-turns, $(0,\ 0),\ (a,\ -a),\ (-a,\ -a)$;

$$x\text{-turns, } (0,\ 0),\ \left(\frac{4a}{9}\sqrt{6},\ -\frac{8a}{9}\right),\ \left(-\frac{4a}{9}\sqrt{6},\ -\frac{8a}{9}\right).$$

Now there appears to be an x-turn and a y-turn at the point $(0, 0)$, and in a certain sense this is evidently true; but we should regard the result as signifying that $(0, 0)$ is a multiple point of *some kind*, since $x = 0$, $y = 0$ satisfy equations (1), (2), and (3).

Let us now determine the character of the point. Dividing (3) by (2), we have

$$\frac{dy}{dx} = \frac{4x^3 + 4axy}{3ay^2 - 2ax^2}.$$

Our object now is to find the value of the slope $\frac{dy}{dx}$ at the multiple point $(0, 0)$. For these values of x and y, $\frac{dy}{dx}$ assumes the form of $\frac{0}{0}$, hence the value required may be obtained by Art. 137.

We see from Ex. 5, Art. 137, that $\frac{dy}{dx} = 0$ and $\pm\sqrt{2}$ at the point (0, 0). Hence the origin (0, 0) is a triple point, the three branches which pass through the point being inclined to the x-axis at the angles 0, $\tan^{-1}\sqrt{2}$ and $\tan^{-1}(-\sqrt{2})$, respectively, as in the figure. See Art. 179.

5. $y^2 = a^2x^2 - x^4$.

x-turns, (0, 0), $(a, 0)$, $(-a, 0)$;

y-turns, (0, 0), $\left(\frac{a}{2}\sqrt{2},\ \pm\frac{1}{2}a^2\right)$, $\left(-\frac{a}{2}\sqrt{2},\ \pm\frac{1}{2}a^2\right)$.

The point (0, 0) is a double point of intersection, since at that point $\frac{dy}{dx} = \pm a$.

6. Examine $y^2(a^2 - x^2) - x^4 = 0$ for multiple points.

At the point (0, 0), $\frac{dy}{dx} = \pm 0$; that is, it has two equal values; hence (0, 0) is an osculating point or a cusp; and since the curve is symmetrical with respect to both axes the point is evidently an osculating point of the first species.

7. Determine the general form of the curve $y^2 = a^2x^3$.

When $x = \infty$, $y = \pm\infty$; hence the curve has two infinite branches, one in the first and one in the fourth quadrant.

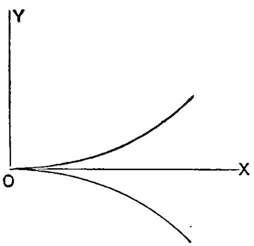

FIG. 36.

When x is negative, y is imaginary; hence the curve does not extend to the left of the y-axis.

When $x = 0$, $y = 0$; hence both branches start from the origin.

At the point (0, 0), $\frac{dy}{dx} = \pm 0$; hence, since the curve is symmetrical with respect to the x-axis, the origin is a cusp of the first species.

Again, since $\frac{d^2y}{dx^2} = \pm \frac{3a}{2\sqrt{x}}$, the upper branch is convex and the lower concave.

8. Examine the curve $(y - x^2)^2 = x^5$, or $y = x^2 \pm x^{\frac{5}{2}}$.

Has two infinite branches, one in the first and one in the fourth quadrant, both starting from the origin. For every positive value of x, y has two real values, both of which are positive as long as $x < 1$, but at the point where $x = 1$ the lower branch crosses the x-axis. The origin is a cusp of the second species.

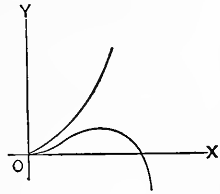

FIG. 37.

178. Tracing Polar Curves. Let $f(r, \theta) = 0$ be the polar equation of the curve.

(*a*) By solving the equation $f(a, \theta) = 0$ for θ, we find the direction of the curve at the point $r = a$. If $a = 0$, the values of θ will be the angles at which the curve cuts the polar axis at the pole.

(*b*) By solving the equation $\frac{dr}{d\theta} = 0$ for θ we find the values of θ for which r is a maximum or minimum, or the r-turns, at which the curve is perpendicular to the radius vector.

9. Trace the curve $r = a \sin 3\theta$, Fig. 38.

(*a*) Making $r = 0$, we have $\sin 3\theta = 0$; hence $\theta = 0, \frac{2}{3}\pi, \frac{4}{3}\pi$, which are the angles at which the curve cuts the polar axis at the pole.

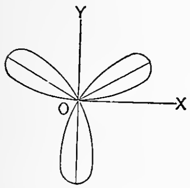

FIG. 38.

(*b*) $\frac{dr}{d\theta} = 3a \cos 3\theta = 0$; hence the values of θ at the r-turns are $\frac{1}{6}\pi$, $\frac{\pi}{2}$, $\frac{5}{6}\pi$, at which points $r = a, -a, +a$, respectively.

Since $\frac{dr}{d\theta} = 3a \cos 3\theta$, r increases from 0 to a, while θ increases from 0 to $\frac{1}{6}\pi$; r decreases from a to $-a$, while θ increases from $\frac{1}{6}\pi$ to $\frac{1}{2}\pi$; r increases from $-a$ to $+a$, while θ increases from $\frac{1}{2}\pi$ to $\frac{5}{6}\pi$; and r decreases from a to 0,

while θ increases from $\frac{5}{6}\pi$ to π. Further revolution of the radius vector would retrace the loops already found.

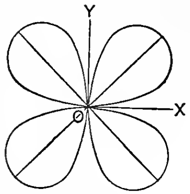

FIG. 39.

10. Trace the curve $r = a \sin 2\theta$, Fig. 39.

From this and the previous example, we infer that the locus of $r = a \sin n\theta$ consists of n loops when n is odd, and $2n$ loops when n is even.

11. Trace the curve $r = a \cos \theta \cos 2\theta$, or $r = a(2 \cos^3 \theta - \cos \theta)$, Fig. 40.

12. Trace the lemniscate $r^2 = a^2 \cos 2\theta$, Fig. 41.

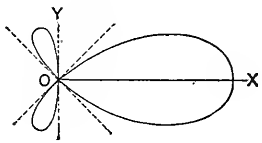

FIG. 40.

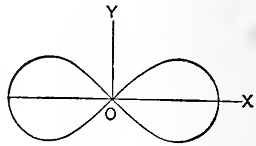

FIG. 41.

179. The character of multiple points in rectangular co-ordinates may often be more easily determined by changing to polar co-ordinates, and applying (a) of Art. 178.

Thus, in Ex. 4, Art. 177, make $y = r \sin \theta$ and $x = r \cos \theta$, divide by r^3, and we have

$$r \cos^4 \theta + 2a \cos^2 \theta \sin \theta - a \sin^3 \theta = 0.$$

Now making $r = 0$, and we have

$$\sin \theta = 0; \quad \text{and} \quad \tan^2 \theta = 2;$$

that is, the angles at which the curve cuts the x-axis at the origin are $\sin^{-1} 0$, $\tan^{-1} \sqrt{2}$, $\tan^{-1} - \sqrt{2}$.

Trace the following curves:

13. The Cissoid, $\quad y^2(2a - x) = x^3$.

14. The Conchoid, $\quad x^2y^2 = (b^2 - y^2)(a + y)^2$.

15. The Witch, $(x^2 + 4a^2)y = 8a^3$.

16. The Lituus, $r\sqrt{\theta} = a$.

17. The Parabola, $r = a \sec^2 \frac{\theta}{2}$.

18. The Curve, $r = a \sin^3 \frac{\theta}{3}$.

19. The Cardioid, $r = a(1 - \cos\theta)$.

20. The Hypocycloid, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

21. Examine $ay^2 = x^3 - bx^2$ for multiple points.

(0, 0) is a conjugate point.

22. Prove that $y^2 = x^5$ and $(y - x)^2 = x^3$ have cusps of the first species at the origin.

23. Prove that $x^4 - 2ax^2y - axy^2 + a^2y^2 = 0$ has a cusp of the second species at the origin.

24. Prove that $x^4 - ax^2y - axy^2 + a^2y^2 = 0$ has a conjugate point at the origin.

25. Prove that the multiple point of $ay^2 = (x - a)^2(x - b)$ at $(a, 0)$ is (1) a conjugate point if $a < b$, (2) a double point if $a > b$, and (3) a cusp if $a = b$.

CHAPTER VIII.

GENERAL DEPENDENT INTEGRATION.

FUNDAMENTAL FORMULAS

180. The differentials in the following twenty-two formulas are the fundamental integrable forms, to one of which we endeavor to reduce every differential that is to be integrated by the dependent process (Art. 51):

1. $\int v^n dv = \dfrac{v^{n+1}}{n+1} + C.$

2. $\int \dfrac{dv}{v} = \log(v) + C,$ or $\log v + \log c = \log cv.$

3. $\int a^v dv = \dfrac{a^v}{\log a} + C.$

4. $\int e^v dv = e^v + C.$

5. $\int \cos v \, dv = \sin v + C.$

6. $\int \sin v \, dv = -\cos v + C.$

7. $\int \sec^2 v \, dv = \tan v + C.$

8. $\int \operatorname{cosec}^2 v \, dv = -\cot v + C.$

9. $\int \sec v \tan v \, dv = \sec v + C.$

10. $\int \operatorname{cosec} v \cot v \, dv = - \operatorname{cosec} v + C.$

11. $\int \tan v \, dv = \log \sec v + C.$

12. $\int \cot v \, dv = \log \sin v + C.$

13. $\int \sec v \, dv = \log (\sec v + \tan v) + C.$

14. $\int \operatorname{cosec} v \, dv = \log (\operatorname{cosec} v - \cot v) + C.$

15. $\int \frac{dv}{\sqrt{1 - v^2}} = \sin^{-1} v + C.$

16. $\int \frac{- dv}{\sqrt{1 - v^2}} = \cos^{-1} v + C.$

17. $\int \frac{dv}{1 + v^2} = \tan^{-1} v + C.$

18. $\int \frac{- dv}{1 + v^2} = \cot^{-1} v + C.$

19. $\int \frac{dv}{v\sqrt{v^2 - 1}} = \sec^{-1} v + C.$

20. $\int \frac{- dv}{v\sqrt{v^2 - 1}} = \operatorname{cosec}^{-1} v + C.$

21. $\int \frac{dv}{\sqrt{2v - v^2}} = \operatorname{vers}^{-1} v + C.$

22. $\int \frac{dv}{\sqrt{v^2 \pm m}} = \log (v + \sqrt{v^2 \pm m}) + C.$

In these formulas v may be the independent variable, or some function of this variable, and the process of integration consists largely in reducing or transforming any given differential into one of the above forms.

REDUCTION AND INTEGRATION OF DIFFERENTIALS.

181. Reduction of Differentials is the process of reducing them to integrable forms, and is effected chiefly (1) by constant multipliers, (2) by decomposing or separating them into their integrable parts, and (3) by substitution.

182. Reduction of Differentials by Constant Multipliers.

PRINCIPLE. The value of any differential of the form $c\int dv$ remains unchanged if dv be multiplied and c be divided by the same constant.

EXAMPLES.

Find the following:

1. $y=\int \frac{ndv}{a+v^2}$. Ans. $\frac{n}{\sqrt{a}}\tan^{-1}\frac{v}{\sqrt{a}}+C$.

We reduce this to the form of 17, Art. 180; thus

$$y=\frac{n}{a}\int\frac{dv}{1+\frac{v^2}{a}}=\frac{n\sqrt{a}}{a}\int\frac{d\left(\frac{v}{\sqrt{a}}\right)}{1+\left(\frac{v}{\sqrt{a}}\right)^2}=\text{Answer.}$$

2. $y=\int\frac{ndv}{\sqrt{a-v^2}}$. (See 15, Art. 180.) Ans. $n\sin^{-1}\frac{v}{\sqrt{a}}+C$.

$$y=\frac{n}{\sqrt{a}}\int\frac{dv}{\sqrt{1-\frac{v^2}{a}}}=\frac{n\sqrt{a}}{\sqrt{a}}\int\frac{d\left(\frac{v}{\sqrt{a}}\right)}{\sqrt{1-\left(\frac{v}{\sqrt{a}}\right)^2}}=\text{Answer.}$$

Let the student compare these results with formulas (13) and (15), page 65, and in a similar manner deduce formulas (14) and (16) on that page.

3. $y = \int \frac{(3x + 3)dx}{x^2 + 2x + 5}$. Ans. $\frac{3}{2} \log (x^2 + 2x + 5) + C$.

$$y = \frac{3}{2}\int \frac{(2x + 2)dx}{x^2 + 2x + 5} = \frac{3}{2}\int \frac{d(x^2 + 2x + 5)}{x^2 + 2x + 5} = \text{Ans.}$$

(Formula 2, Art. 180.)

4. $\int \frac{5x^2dx}{7x^3 + 4}$. $\frac{5}{21} \log (7x + 4) + C$.

5. $\int \frac{dx}{\sqrt{2 - 9x^2}}$. $\frac{1}{3} \sin^{-1} \frac{3x}{\sqrt{2}} + C$.

6. $\int \frac{3dx}{\sqrt{6x - 9x^2}}$. $\text{vers}^{-1}\, 3x + C$.

7. $\int \frac{3dx}{2 + 7x^2}$. $\frac{3}{\sqrt{14}} \tan^{-1} x\sqrt{\frac{7}{2}} + C$.

8. $\int \frac{xdx}{1 + x^4}$. $\frac{1}{2} \tan^{-1} x^2 + C$.

9. $\int \frac{dx}{1 + 5x^2}$. $\frac{1}{\sqrt{5}} \tan^{-1} x\sqrt{5} + C$.

10. $\int \frac{dx}{2 + 3x^2}$. $\frac{1}{\sqrt{6}} \tan^{-1} x\sqrt{\frac{3}{2}} + C$.

11. $\int \frac{dx}{x\sqrt{2x^2 - 3}}$. $\frac{1}{\sqrt{3}} \sec^{-1} x\sqrt{\frac{2}{3}} + C$.

12. $\int \frac{3dx}{\sqrt{9x - 4x^2}}$. $\frac{3}{2} \text{vers}^{-1} \frac{8x}{9} + C$.

13. $\int \frac{- dx}{\sqrt{3 - 2x^2}}$. $\frac{1}{\sqrt{2}} \cos^{-1} x\sqrt{\frac{2}{3}} + C$.

14. $\int \frac{dx}{x^2 + 16}$. $\frac{1}{4} \tan^{-1} \frac{x}{4} + C$.

15. $\displaystyle\int \frac{5dx}{x\sqrt{3x^2-5}}$. $\qquad \sqrt{5}\sec^{-1} x\sqrt{\tfrac{3}{5}} + C.$

16. $\displaystyle\int \frac{dx}{\sqrt{5x^4-3x^2}}$. $\qquad \tfrac{1}{3}\sqrt{3}\sec^{-1} x\sqrt{\tfrac{5}{3}} + C.$

Some of the preceding examples may be conveniently solved by formulas (19) and (21), page 65.

REDUCTION OF DIFFERENTIALS BY DECOMPOSITION.

183. The process of reducing differentials to integrable forms consists largely in separating them into their integrable parts.

184. Elementary Differentials. In these the necessary reductions are effected by the simple operations of algebra.

EXAMPLES.

Find the following:

1. $\displaystyle\int \frac{(3x+5)}{4x^2+1}dx$. $\qquad \tfrac{3}{8}\log(4x^2+1) + \tfrac{5}{2}\tan^{-1}(2x) + C.$

$$\int \frac{(3x+5)}{4x^2+1}dx = \int \frac{3xdx}{4x^2+1} + \int \frac{5dx}{1+4x^2}.$$

2. $\displaystyle\int \frac{(x-3)}{\sqrt{1-x^2}}dx$. $\qquad -(1-x^2)^{\frac{1}{2}} - 3\sin^{-1} x + C.$

3. $\displaystyle\int \frac{(2-5x)}{\sqrt{4x-2x^2}}dx$. $\qquad \dfrac{5}{\sqrt{2}}(2x-x^2)^{\frac{1}{2}} - \dfrac{3}{\sqrt{2}}\operatorname{vers}^{-1} x + C.$

4. $\displaystyle\int \frac{\sqrt{x^2-1}}{x}dx$. $\qquad \left[\dfrac{\sqrt{x^2-1}}{x} = \dfrac{x^2-1}{x\sqrt{x^2-1}}.\right]$

$\qquad (x^2-1)^{\frac{1}{2}} + \operatorname{cosec}^{-1} x + C$

5. $\displaystyle\int \frac{\sqrt{a+x}}{\sqrt{a-x}}\,dx.$ $\quad\left[\dfrac{\sqrt{a+x}}{\sqrt{a-x}} = \dfrac{a+x}{\sqrt{a^2-x^2}}.\right]$

$$\sin^{-1}\frac{x}{a} - \sqrt{a^2-x^2} + C.$$

6. $\displaystyle\int \left(\frac{x+1}{x}\right)^3 dx.$ $\qquad x + 3\log x - \dfrac{3}{x} - \dfrac{1}{2x^2} + C.$

7. $\displaystyle\int \left(\frac{x^3-1}{x-1}\right) dx.$ $\qquad \dfrac{x^3}{3} + \dfrac{x^2}{2} + x + C.$

8. $\displaystyle\int \frac{x^4}{x+1}\,dx.$ $\qquad \dfrac{x^4}{4} - \dfrac{x^3}{3} + \dfrac{x^2}{2} - x + \log(1+x) + C.$

185. Trigonometric Differentials.—In reducing these we use the elementary formulas of Trigonometry, such as

$$\sin^2 x + \cos^2 x = 1, \quad \sin 2x = 2\sin x\cos x,$$
$$\cos 2x = \cos^2 x - \sin^2 x, \text{ etc.}$$

EXAMPLES.

Find:

1. $\displaystyle\int \sin^3 x\,dx.$ $\qquad -\cos x + \tfrac{1}{3}\cos^3 x + C.$

$$\sin^3 x = \sin x\,(1-\cos^2 x) = \sin x - (\cos^2 x)\sin x.$$

$$\therefore \quad \int (\sin^3 x)dx = \int \sin x\,dx + \int (\cos x)^2 d(\cos x).$$

2. $\displaystyle\int \sin^5 x\,dx.$ $\quad\left[= \displaystyle\int (1-\cos^2 x)^2 \sin x\,dx.\right]$

$$-\cos x + \tfrac{2}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + C.$$

3. $\displaystyle\int \sin^7 x\,dx.$ $\qquad -\cos x + \cos^3 x - \tfrac{3}{5}\cos^5 x + \tfrac{1}{7}\cos^7 x + C.$

4. $\displaystyle\int \cos^3 x\,dx.$ $\qquad \sin x - \tfrac{1}{3}\sin^3 x + C.$

$$\cos^3 x = \cos x(1-\sin^2 x).$$

5. $\int \cos^5 x\, dx.$ $\qquad\qquad \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$

In like manner $\int \cos^m x\, dx$ and $\int \sin^m x\, dx$ can be found where m is any odd positive integer.

6. $\int \sin^4 x \cos^3 x\, dx.$ $\qquad\qquad \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C.$

$$\sin^4 x \cos^3 x = \sin^4 x(1 - \sin^2 x)\cos x.$$

In a similar manner $\int \sin^m x \cos^n x\, dx$ may be found when either m or n is any odd positive integer.

7. $\int \sin^3 x \cos^4 x\, dx.$ $\qquad\qquad -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C.$

8. $\int \sin^2 x \cos^7 x\, dx.$ $\qquad\qquad \frac{1}{3}\sin^3 x - \frac{3}{5}\sin^5 x + \frac{3}{7}\sin^7 x - \frac{1}{9}\sin^9 x + C.$

9. $\int \sin^3 x \sqrt{\cos x}\, dx.$ $\qquad\qquad -\frac{2}{3}(\cos x)^{\frac{3}{2}} + \frac{2}{7}(\cos x)^{\frac{7}{2}} + C.$

10. $\int \cos^2 x\, dx.$ $\qquad\qquad \frac{x}{2} + \frac{1}{4}\sin 2x + C.$

$$\cos^2 x = \tfrac{1}{2} + \tfrac{1}{2}\cos 2x.$$

11. $\int \sin^2 x\, dx.$ $\qquad\qquad \frac{x}{2} - \frac{1}{4}\sin 2x + C,$

or $\frac{x}{2} - \frac{1}{2}\sin x \cos x + C.$

12. $\int \frac{dx}{\sin x \cos x}.$ $\left[= \int \frac{\sec^2 x}{\tan x} dx.\right]$ $\qquad\qquad \log \tan x + C.$

13. $\int \frac{dx}{\sin^2 x \cos^2 x}.$ $\qquad\qquad \tan x - \cot x + C.$

$$\frac{1}{\sin^2 x \cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}.$$

14. $\int \frac{\sin^3 x\,dx}{\cos^2 x}$. $\qquad \sec x + \cos x + C.$

15. $\int \frac{\sin^2 x}{\cos^6 x} dx$. $\qquad \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C.$

$$\frac{\sin^2 x}{\cos^6 x} = \tan^2 x \sec^4 x = \tan^2 x(1 + \tan^2 x)\sec^2 x.$$

In like manner $\frac{\sin^m x}{\cos^n x} dx$ or $\frac{\cos^m x}{\sin^n x} dx$ may be integrated whenever $m - n$ is even and negative.

16. $\int \frac{dx}{\cos^4 x}$. $\qquad \tan x + \frac{1}{3}\tan^3 x + C.$

17. $\int \frac{\cos^2 x\,dx}{\sin^4 x}$. $\qquad -\frac{1}{3}\cot^3 x + C.$

186. Trigonometric differentials can often be more conveniently integrated as indicated by the following solutions.

18. $\int \cos^4 x \sin^3 x\,dx$. $\qquad -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C.$

Make $\cos x = y$; then $\sin x = \sqrt{1 - y^2}$ and $dx = -\frac{dy}{\sqrt{1-y^2}}$.

$$\therefore \quad \int \cos^4 x \sin^3 x\,dx = \int -y^4(1 - y^2)dy = -\tfrac{1}{5}y^5 + \tfrac{1}{7}y^7 + C.$$

19. $\int \tan^5 x\,dx$. $\qquad \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \log(\sec x) + C.$

Make $\tan x = y$; then $dx = \frac{dy}{y^2 + 1}$.

$$\therefore \quad \int \tan^5 x\,dx = \int \frac{y^5}{y^2+1} dy = \int \left(y^3 - y + \frac{y}{y^2+1}\right) dy$$

$$= \tfrac{1}{4}y^4 - \tfrac{1}{2}y^2 + \tfrac{1}{2}\log(y^2 + 1) + C.$$

This method combined with that of Arts. 212 to 215 affords a complete solution of rational trigonometric differentials.

187. Rational Fractions. A fraction whose terms involve only a finite number of positive and integral powers of the variable is called a **Rational Fraction**; as

$$dy = \frac{x^4 - 2x^3 - 13x^2 + 17}{x^2 + 3x + 2} dx.$$

To separate this fraction into its integrable parts, we first divide the numerator by the denominator and obtain

$$dy = (x^2 - 5x)dx + \frac{10x + 17}{x^2 + 3x + 2} dx.$$

Again, by separating the fractional part of this quotient into two parts (its "partial" fractions), we obtain

$$\frac{10x + 17}{x^2 + 3x + 2} dx = \frac{7dx}{x + 1} + \frac{3dx}{x + 2}.$$

$$\therefore \qquad dy = (x^2 - 5x)dx + \frac{7dx}{x + 1} + \frac{3dx}{x + 2}.$$

$$\therefore \qquad y = \int (x^2 - 5x)dx + 7\int \frac{dx}{x + 1} + 3\int \frac{dx}{x + 2};$$

or
$$y = \frac{x^3}{3} - \frac{5x^2}{2} + 7 \log (x + 1) + 3 \log (x + 2) + C.$$

That is,
$$y = \frac{2x^3 - 15x^2}{6} + \log (x + 1)^7 (x + 2)^3 + C.$$

The first step in the above and similar operations is very simple, and it is our present purpose to show how the second step, the separation and integration of fractions whose denominators contain a higher power of x than the numerator, may be effected; and to render the process as simple as possible we shall apply it to particular examples in each of the four cases that may occur.

188. CASE I. *When the simple factors of the denominator are real and unequal.*

EXAMPLES.

1. Integrate $$dy = \frac{(x+1)dx}{x^3 + 6x^2 + 8x}.$$

The roots of $x^3 + 6x^2 + 8x = 0$, are 0, -2, and -4; hence the factors of $x^3 + 6x^2 + 8x$ are x, $x+2$, and $x+4$.

Assume $$\frac{x+1}{x^3+6x^2+8x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x+4}. \quad . \quad . \quad (1)$$

Clearing (1) of fractions, we have

$$x + 1 = A(x+2)(x+4) + B(x)(x+4) + C(x)(x+2), \quad (2)$$

or $$x+1 = (A+B+C)x^2 + (6A + 4B + 2C)x + 8A.$$

Equating the coefficients of the like powers of x, we have

$$A + B + C = 0, \quad 6A + 4B + 2C = 1, \quad 8A = 1.$$

Solving these equations, we find $A = \frac{1}{8}$, $B = \frac{1}{4}$, and $C = -\frac{3}{8}$. Substituting these values in (1), we have

$$\frac{x+1}{x^3+6x^2+8x} = \frac{1}{8x} + \frac{1}{4(x+2)} - \frac{3}{8(x+4)}.$$

$$\therefore\ dy = \frac{dx}{8x} + \frac{dx}{4(x+2)} - \frac{3dx}{8(x+4)}.$$

$$\therefore\ y = \tfrac{1}{8}\int\frac{dx}{x} + \tfrac{1}{4}\int\frac{dx}{x+2} - \tfrac{3}{8}\int\frac{dx}{x+4}$$

$$= \tfrac{1}{8}\log x + \tfrac{2}{8}\log(x+2) - \tfrac{3}{8}\log(x+4) + \tfrac{1}{8}\log c.$$

$$\therefore\ y = \log\sqrt[8]{\frac{cx(x+2)^2}{(x+4)^3}}.$$

The values of A, B, and C may be obtained from (2), thus:

Making $x = 0$, we have $1 = 8A$; $\therefore A = \frac{1}{8}$.

Making $x = -2$, we have $-1 = -4B$; $\therefore B = \frac{1}{4}$.

Making $x = -4$, we have $-3 = 8C$; $\therefore C = -\frac{3}{8}$.

Principle. *In this case, to every factor of the denominator, as* $x - a$, *there corresponds a partial fraction of the form* $\dfrac{A}{x-a}$.

Find the following:

2. $\displaystyle\int \frac{3dx}{x^2 - 4}$. $\qquad \log \left(\frac{x-2}{x+2}\right)^{\frac{3}{4}} + C$.

3. $\displaystyle\int \frac{adx}{x^2 - a^2}$. $\qquad \log \sqrt{\frac{x-a}{x+a}} + C$.

4. $\displaystyle\int \frac{(5x+1)dx}{x^2 + x - 2}$. $\qquad \log\,[(x-1)^2(x+2)^3c]$.

5. $\displaystyle\int \frac{(x^2 + x - 1)dx}{x^3 + x^2 - 6x}$. $\qquad \log \sqrt[6]{x(x-2)^3(x+3)^2} + C$.

6. $\displaystyle\int \frac{x^5 + x^4 - 8}{x^3 - 4x}dx$. $\qquad \dfrac{x^3}{3} + \dfrac{x^2}{2} + 4x + \log\dfrac{x^2(x-2)^5}{(x+2)^3} + C$.

189. Case II. *When some of the simple factors of the denominator are real and equal.*

EXAMPLES.

1. Integrate $\qquad dy = \dfrac{(x^2 + x)dx}{(x-2)^2(x-1)}$.

Assume $\qquad \dfrac{x^2 + x}{(x-2)^2(x-1)} = \dfrac{A}{(x-2)^2} + \dfrac{B}{x-2} + \dfrac{C}{x-1}$.

Clearing of fractions, we have

$$x^2 + x = A(x-1) + B(x-2)(x-1) + C(x-2)^2.$$

Making $x = 2$, we have $6 = A$; $\quad \therefore\ A = 6$.

Making $x = 1$, we have $2 = C$; $\quad \therefore\ C = 2$.

Making $x = 0$, we have $0 = -A + 2B + 4C$; $\quad \therefore\ B = -1$.

$$\therefore \quad \frac{x^2 + x}{(x-2)^2(x-1)} = \frac{6}{(x-2)^2} - \frac{1}{x-2} + \frac{2}{x-1}.$$

$$\therefore \quad dy = \frac{6dx}{(x-2)^2} - \frac{dx}{x-2} + \frac{2dx}{x-1}.$$

$$\therefore \quad y = 6\int \frac{dx}{(x-2)^2} - \int \frac{dx}{x-2} + 2\int \frac{dx}{x-1}$$

$$= -\frac{6}{x-2} - \log(x-2) + 2\log(x-1) + C$$

$$= \log\frac{(x-1)^2}{(x-2)} - \frac{6}{x-2} + C.$$

Principle. *In this case, to every factor of the form* $(x-a)^n$ *there corresponds a series of* n *partial fractions of the form*

$$\frac{A}{(x-a)^n},\ \frac{B}{(x-a)^{n-1}},\ \ldots\ \frac{K}{x-a}.$$

Find the following:

2. $\displaystyle\int \frac{(3x-1)dx}{(x-3)^2}$. $\qquad \displaystyle 3\log(x-3) - \frac{8}{x-3} + C.$

3. $\displaystyle\int \frac{(9x^2 + 9x - 128)dx}{(x-3)^2(x+1)}$.

$$\frac{5}{x-3} + 17\log(x-3) - 8\log(x+1) + C.$$

4. $\displaystyle\int \frac{x^2dx}{x^3 + 5x^2 + 8x + 4}$. $\qquad \displaystyle \frac{4}{x+2} + \log(x+1) + C.$

5. $\displaystyle\int \frac{dx}{(x+2)^2(x+3)^2}$. $\qquad \displaystyle \log\left(\frac{x+3}{x+2}\right)^2 - \frac{2x+5}{(x+2)(x+3)} + C.$

6. $\displaystyle\int \frac{3x+2}{x(x+1)^3}dx$. $\qquad \displaystyle \frac{4x+3}{2(x+1)^2} + 2\log\left(\frac{x}{x+1}\right) + C.$

190. Case III. *When some of the simple factors of the denominator are imaginary and unequal.*

EXAMPLES.

1. Integrate $\qquad dy = \dfrac{xdx}{(x+1)(x^2+4)}.$

Here the two simple factors of x^2+4 are $x + 2\sqrt{-1}$ and $x - 2\sqrt{-1}$; we may take these factors and proceed as in Case I, but the integrals obtained would involve the logarithms of imaginaries; to obviate this, we assume

$$\frac{x}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}.$$

Clearing of fractions, we have

$$x = A(x^2+4) + (Bx+C)(x+1). \quad . \quad . \quad . \quad (1)$$

Differentiating (1), we have

$$1 = 2Ax + B(x+1) + Bx + C. \quad . \quad . \quad . \quad (2)$$

In (1) making $x = -1$, we have $A = -\frac{1}{5}$.

In (1) making $x = 0$, we have $C = -4A = +\frac{4}{5}$.

In (2) making $x = 0$, we have $B = 1 - C = \frac{1}{5}$.

$$\therefore\ dy = -\tfrac{1}{5}\left(\frac{dx}{x+1}\right) + \tfrac{1}{5}\left(\frac{x+4}{x^2+4}\right)dx$$

$$= -\tfrac{1}{5}\left(\frac{dx}{x+1}\right) + \tfrac{1}{5}\left(\frac{xdx}{x^2+4}\right) + \tfrac{4}{5}\left(\frac{dx}{x^2+4}\right).$$

$$\therefore\ \ y = -\tfrac{1}{5}\log(x+1) + \tfrac{1}{10}\log(x^2+4) + \tfrac{2}{5}\tan^{-1}\frac{x}{2} + C$$

$$= \log\sqrt[10]{\frac{x^2+4}{(x+1)^2}} + \tfrac{2}{5}\tan^{-1}\frac{x}{2} + C.$$

Principle. *In this case, to every factor of the denominator of the form* $(x-a)^2+b^2$ *there corresponds a partial fraction of the form* $\dfrac{Ax+B}{(x-a)^2+b^2}.$

2. $\int \frac{x^2dx}{1-x^4}$. $\qquad \frac{1}{4}\log\frac{1+x}{1-x} - \frac{1}{2}\tan^{-1}x + C.$

3. $\int \frac{dx}{(x^2+1)(x^2+4)}$. $\qquad \frac{1}{3}\tan^{-1}x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C.$

4. $\int \frac{x^2dx}{x^4+x^2-2}$. $\qquad \frac{1}{6}\log\left(\frac{x-1}{x+1}\right) + \frac{\sqrt{2}}{3}\tan^{-1}\frac{x}{\sqrt{2}} + C.$

5. $\int \frac{x^3-1}{x^3+3x}dx$. $\qquad x + \frac{1}{6}\log\frac{x^2+3}{x^2} - \sqrt{3}\tan^{-1}\frac{x}{\sqrt{3}} + C.$

6. $\int \frac{4dx}{x^4+1}$. $\qquad \frac{1}{\sqrt{2}}\log\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1} + \sqrt{2}\tan^{-1}\frac{x\sqrt{2}}{1-x^2} + C.$

191. CASE IV. *When some of the simple factors of the denominator are imaginary and equal.*

EXAMPLE.

1. Integrate $\quad dy = \frac{dx}{(x^2+3)^2(x-1)}$.

Assume $\quad \frac{1}{(x^2+3)^2(x-1)} = \frac{Ax+B}{(x^2+3)^2} + \frac{Cx+D}{x^2+3} + \frac{E}{x-1}$.

Clearing of fractions, we have

$$1 = (Ax+B)(x-1) + (Cx+D)(x^2+3)(x-1) + E(x^2+3)^2. \quad (1)$$

Whence $\quad A = -\frac{1}{4}, \quad B = -\frac{1}{4}, \quad C = -\frac{1}{16},$

$$D = -\frac{1}{16}, \quad E = \frac{1}{16}.$$

$$\therefore \quad dy = -\frac{1}{4}\frac{x+1}{(x^2+3)^2}dx - \frac{1}{16}\frac{(x+1)}{x^2+3} + \frac{1}{16}\frac{dx}{x-1}.$$

$$\therefore \quad y = \frac{1}{8(x^2+3)} - \frac{1}{32}\log(x^2+3) - \frac{1}{16\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}$$

$$+ \frac{1}{16}\log(x-1) - \frac{1}{4}\int\frac{dx}{(x^2+3)^2}.$$

The integration of differentials of the form $\frac{dx}{(ax^2+b)^m}$ may be more conveniently obtained by Art. 211 or 215.

REDUCTION BY SUBSTITUTION.

192. Irrational Differentials. To integrate an irrational differential which is not of one of the known integrable forms, we first rationalize it, and then proceed according to the previous methods.

To show, in a simple manner, how rationalization is to be effected, we shall apply the process to a few particular examples.

EXAMPLES.

Find the following:

1. $y = \int \frac{(2\sqrt{x}+3)dx}{2\sqrt{x}(x+3\sqrt{x}+5)}$.

Make $x = z^2$; $\therefore dx = 2z\,dz$.

$$y = \int \frac{(2z+3)2z\,dz}{2z(z^2+3z+5)} = \log(z^2+3z+5) = \log(x+3\sqrt{x}+5).$$

2. $\int \frac{(2\sqrt{x}+1)dx}{4\sqrt{x^2+x\sqrt{x}}}$. $\qquad \sqrt{x+\sqrt{x}}+C.$

3. $\int \frac{dx}{2(x^{\frac{1}{2}}+x^{\frac{3}{2}})}$. $\qquad \tan^{-1}\sqrt{x}+C.$

4. $\int \frac{dx}{3\sqrt[3]{x^2}\sqrt[2]{1-x^{\frac{2}{3}}}}$. [Make $x = z^3$.] $\qquad \sin^{-1}\sqrt[3]{x}+C.$

5. $\int \frac{dx}{\sqrt{x}(1-x)}$. $\qquad \log\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)+C.$

6. $\int \frac{dx}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}$. [Make $x = z^6$.]

$$2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\log(1+\sqrt[6]{x})+C.$$

7. $\int \frac{(x-1)dx}{2(x-4)\sqrt{x}}$. $\qquad \sqrt{x}+\frac{3}{4}\log\frac{\sqrt{x}-2}{\sqrt{x}+2}+C.$

193. When $a + bx$ is the only part having a fractional exponent.

Assume $a + bx = z^n$, where n is the least common multiple of the denominators of all the fractional exponents; then the values of x, dx, and each of the surds, will be rational in terms of z.

EXAMPLES.

Find

1. $y = \int x^2 \sqrt{1+x}\,dx.$

Assume $1 + x = z^2$; then $\sqrt{1+x} = z$. . . . (1)

Also $x = z^2 - 1$, $x^2 = (z^2 - 1)^2$, . (2)

$dx = 2z\,dz$. . . (3)

Multiplying (1), (2) and (3) together, we have

$$\int x^2 \sqrt{1+x}\,dx = \int 2z^2(z^2 - 1)^2 dz$$
$$= \tfrac{2}{7}z^7 - \tfrac{4}{5}z^5 + \tfrac{2}{3}z^3 + C$$
$$= \tfrac{2}{7}(1+x)^{\frac{7}{2}} - \tfrac{4}{5}(1+x)^{\frac{5}{2}} + \tfrac{2}{3}(1+x)^{\frac{3}{2}} + C.$$

2. $\int \frac{x\,dx}{\sqrt{1+x}}.$ $\qquad \frac{2(x-2)\sqrt{1+x}}{3} + C.$

3. $\int \frac{dx}{x\sqrt{1+x}}.$ $\qquad \log\left(\frac{\sqrt{1+x} - 1}{\sqrt{1+x} + 1}\right) + C.$

4. $\int x(a+x)^{\frac{1}{3}}dx.$ $\qquad \frac{3}{28}(a+x)^{\frac{4}{3}}(4x - 3a) + C.$

194. When $\sqrt{a + bx + x^2}$ or $\sqrt{a + bx - x^2}$ is the only surd involved.

A differential containing no surd except $\sqrt{a + bx + x^2}$ can be rationalized by assuming $\sqrt{a + bx + x^2} = z - x$; and one containing no surd except $\sqrt{a + bx - x^2}$ can be rationalized by

assuming $\sqrt{a + bx - x^2} = (x - r)z$, where r is one of the roots of $a + bx - x^2 = 0$.

The process is illustrated in integrating the following important differentials (see Ex. 30, 31, page 67).

1. Find $y = \int \frac{dx}{\sqrt{a + bx + x^2}}$.

Assume $\sqrt{a + bx + x^2} = z - x$; then

$$a + bx = z^2 - 2zx, \quad x = \frac{z^2 - a}{2z + b},$$

$$dx = \frac{2(z^2 + bz + a)dz}{(2z + b)^2}, \quad \dots \dots \quad (1)$$

$$\frac{1}{\sqrt{a + bx + x^2}} = \frac{2z + b}{z^2 + bz + a}. \quad \dots \dots \quad (2)$$

$$(1) \times (2), \quad \int \frac{dx}{\sqrt{a + bx + x^2}} = \int \frac{2dz}{2z + b}$$

$$= \log(2z + b) + C.$$

$$\therefore \quad \int \frac{dx}{\sqrt{a + bx + x^2}} = \log(2x + b + 2\sqrt{a + bx + x^2}) + C.$$

When $b = 0$,

$$\int \frac{dx}{\sqrt{a + x^2}} = \log(2x + 2\sqrt{a + x^2}) + C.$$

2. Find $$y = \int \frac{dx}{\sqrt{a + bx - x^2}}.$$

Represent the factors of $a + bx - x^2 = 0$ by $x - r$ and $r' - x$, and assume

$$\sqrt{a + bx - x^2} = \sqrt{(x - r)(r' - x)} = (x - r)z,$$

then $\qquad r' - x = (x - r)z^2, \qquad x = \dfrac{rz^2 + r'}{z^2 + 1},$

$$dx = \frac{2(r - r')z\,dz}{(z^2 + 1)^2}, \;..\; (1) \qquad \frac{1}{\sqrt{a + bx - x^2}} = \frac{z^2 + 1}{(r' - r)z}. \;..\; (2)$$

(1) × (2), $\qquad \displaystyle\int \frac{dx}{\sqrt{a + bx - x^2}} = -2\int \frac{dz}{1 + z^2}$

$$= -2 \tan^{-1} z + C.$$

$$\therefore \qquad \int \frac{dx}{\sqrt{a + bx - x^2}} = -2 \tan^{-1} \sqrt{\frac{r' - x}{x - r}} + C.$$

When $b = 0$, $\quad r = +\sqrt{a}$, $\quad$ and $\quad r' = -\sqrt{a}$, $\quad$ we have

$$\int \frac{dx}{\sqrt{a - x^2}} = -2 \tan^{-1} \sqrt{\frac{\sqrt{a} + x}{\sqrt{a} - x}} + C.$$

3. $\displaystyle\int \frac{dx}{x\sqrt{2 + x - x^2}}. \qquad \frac{1}{\sqrt{2}} \log\left(\frac{\sqrt{2 + 2x} - \sqrt{2 - x}}{\sqrt{2 + 2x} + \sqrt{2 - x}}\right) + C.$

Assume $\quad \sqrt{2 + x - x^2} = \sqrt{(2 - x)(1 + x)} = (2 - x)z$; then

$$x = \frac{2z^2 - 1}{z^2 + 1}, \quad dx = \frac{6z\,dz}{(z^2 + 1)^2}, \quad \text{and} \quad \sqrt{2 + x - x^2} = \frac{3z}{z^2 + 1}.$$

$$\therefore \qquad \int \frac{dx}{x\sqrt{2 + x - x^2}} = \int \frac{2dz}{2z^2 - 1} = \frac{1}{\sqrt{2}} \log \frac{z\sqrt{2} - 1}{z\sqrt{2} + 1}.$$

4. $\displaystyle\int \frac{dx}{x\sqrt{x^2 + 2x - 1}}. \qquad 2 \tan^{-1}\left(x + \sqrt{x^2 + 2x - 1}\right) + C.$

5. $\displaystyle\int \frac{x^2dx}{\sqrt{3 + 2x - x^2}}. \qquad 3 \sin^{-1} \frac{x - 1}{2} - \frac{(x + 3)\sqrt{3 + 2x - x^2}}{2}.$

195. Binomial Differentials. Differentials of the form

$$x^m(a + bx^n)^{\frac{r}{s}}dx,$$

where m, n, r, and s represent any positive or negative integers, are called binomial differentials.

196. To determine the conditions under which a binomial differential is integrable.

I. When $\frac{r}{s}$ is a positive integer the binomial factor can be developed in a finite number of terms, and the differential exactly integrated; and when $\frac{r}{s}$ is a negative integer the differential is a rational fraction whose integral can be obtained by the method of Art. 187, 212, 214, or 215.

II. Assume $\quad a + bx^n = z^s; \quad \therefore \ (a + bx^n)^{\frac{r}{s}} = z^r, \quad . \quad . \quad (1)$

$$x = \left(\frac{z^s - a}{b}\right)^{\frac{1}{n}}, \qquad x^m = \left(\frac{z^s - a}{b}\right)^{\frac{m}{n}}, \quad . \quad . \quad . \quad (2)$$

and

$$dx = \frac{s}{bn} z^{s-1}\left(\frac{z^s - a}{b}\right)^{\frac{1}{n} - 1} dz. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Multiplying (1), (2) and (3) together, we have

$$x^m(a + bx^n)^{\frac{r}{s}}\,dx = \frac{s}{bn}\, z^{r+s-1}\left(\frac{z^s - a}{b}\right)^{\frac{m+1}{n} - 1} dz. \quad . \quad . \quad (4)$$

The second member of (4), and therefore the first, is integrable when $\frac{m+1}{n}$ is a positive or negative integer, by Case I.

III. Assume $\quad a + bx^n = z^s x^n; \quad \therefore \ x^n = a(z^s - b)^{-1},$

$$x = a^{\frac{1}{n}}(z^s - b)^{-\frac{1}{n}}, \qquad x^m = a^{\frac{m}{n}}(z^s - b)^{-\frac{m}{n}}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$a + bx^n = \frac{az^s}{z^s - b}, \qquad (a + bx^n)^{\frac{r}{s}} = a^{\frac{r}{s}}(z^s - b)^{-\frac{r}{s}}z^r dz, \quad . \quad . \quad . \quad (2)$$

and $$dx = -\frac{s}{n}a^{\frac{1}{n}}z^{s-1}(z^s - b)^{-\frac{1}{n}-1}dz. \quad (3)$$

Multiplying (1), (2) and (3) together, we have

$$x^m(a + bx^n)^{\frac{r}{s}}dx = -\frac{s}{n}a^{\frac{m+1}{n}+\frac{r}{s}}(z^s - b)^{-\left(\frac{m+1}{n}+\frac{r}{s}+1\right)}z^{r+s-1}dz. \quad (4)$$

By Case I, the second member of (4) is integrable when $\frac{m+1}{n} + \frac{r}{s}$ is a positive or negative integer.

Hence, $x^m(a + bx^n)^{\frac{r}{s}}dx$ can be integrated by rationalization:

I. *When* $\frac{m+1}{n}$ *is an integer or* 0, *by assuming* $a + bx^n = z^s$.

II. *When* $\frac{m+1}{n} + \frac{r}{s}$ *is an integer or* 0, *by assuming*

$$a + bx^n = z^s x^n.$$

When the differential reduces to a rational fraction, which is the case when $\frac{m+1}{n} + 1$ is a negative, or $\frac{m+1}{n} + \frac{r}{s} + 1$ a positive, integer, it is less laborious to integrate by a method to be subsequently given.

EXAMPLES.

1. Find $\int x^5(1 + x^2)^{\frac{1}{2}}dx$.

Here $\frac{m+1}{n} = \frac{5+1}{2} = 3$, an integer, and $s = 2$; hence we assume

$$1 + x^2 = z^2; \qquad \therefore (1 + x^2)^{\frac{1}{2}} = z, \quad . \quad . \quad . \quad . \quad (1)$$

$$x^2 = z^2 - 1, \qquad x^6 = (z^2 + 1)^3. \quad . \quad . \quad . \quad . \quad (2)$$

Differentiating (2),

$$6x^5dx = 6(z^2 + 1)^2z\,dz. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Multiplying (1) and (3) and dividing by 6, we have

$$\int x^5(1+x^2)^{\frac{1}{2}}dx = \int (z^2+1)^2 z^2 dz$$

$$= \int (z^6 + 2z^4 + z^2)dz$$

$$= \tfrac{1}{7}z^7 + \tfrac{2}{5}z^5 + \tfrac{1}{3}z^3 + C$$

$$= \tfrac{1}{7}(1+x^2)^{\frac{7}{2}} + \tfrac{2}{5}(1+x^2)^{\frac{5}{2}} + \tfrac{1}{3}(1+x^2)^{\frac{3}{2}} + C.$$

Find:

2. $\int x^3(1+x^2)^{\frac{1}{2}}dx.$ $\qquad (1+x^2)^{\frac{3}{2}}\left(\frac{3x^2-2}{15}\right) + C.$

3. $\int x^5(1+x^2)^{\frac{2}{3}}dx.$

$\qquad \frac{3}{22}(1+x^2)^{\frac{11}{3}} - \frac{3}{8}(1+x^2)^{\frac{8}{3}} + \frac{3}{10}(1+x^2)^{\frac{5}{3}} + C.$

4. $\int x^3(1+x^2)^{-\frac{1}{2}}dx.$ $\qquad (1+x^2)^{\frac{1}{2}}\left(\frac{x^2-2}{3}\right) + C.$

5. $\int x^{-2}(1+x^2)^{-\frac{3}{2}}dx.$ $\qquad -\frac{2x^2+1}{x\sqrt{1+x^2}} + C.$

Here

$$\frac{m+1}{n} = \frac{-2+1}{2} = -\frac{1}{2}; \quad \text{and} \quad \frac{m+1}{n} + \frac{r}{s} = -\frac{1}{2} - \frac{3}{2} = -2,$$

an integer; hence we assume

$$1 + x^2 = x^2z^2; \quad \therefore\ (1+x^2)^{-\frac{3}{2}} = \frac{(z^2-1)^{\frac{3}{2}}}{z^3}; \quad . \quad . \quad . \quad (1)$$

$$x^{-2} = z^2 - 1; \quad . \quad . \quad . \quad . \quad (2)$$

$$dx = -\frac{zdz}{(z^2-1)^{\frac{3}{2}}}. \quad . \quad . \quad (3)$$

Multiplying (1), (2), and (3) together, we have

$$\int x^{-2}(1+x^2)^{-\frac{3}{2}}dx = \int -\left(1-\frac{1}{z^2}\right)dz$$

$$= -\left(z+\frac{1}{z}\right) + C,$$

where $z = \frac{1}{x}\sqrt{1+x^2}$.

6. $\int (1+x^2)^{-\frac{3}{2}}dx.$ $[m=0.]$ $\qquad \dfrac{x}{\sqrt{1+x^2}} + C.$

7. $\int x^{-4}(1-2x^2)^{-\frac{1}{2}}dx.$ $\qquad -\dfrac{(1+4x^2)(1-2x^2)^{\frac{1}{2}}}{3x^3} + C.$

8. $\int x^{-2}(a+x^3)^{-\frac{5}{3}}dx.$ $\qquad -\dfrac{3x^3+2a}{2a^2x(a+x^3)^{\frac{2}{3}}} + C.$

INTEGRATION BY PARTS.

197. Integrating both members of

$$d(uv) = udv + vdu$$

and transposing, we have

$$\int udv = uv - \int vdu, \quad \ldots \ldots \quad (A)$$

which is the formula for integration by parts. It reduces the integration of udv to that of vdu, and by its application many differentials can be reduced to one of the elementary forms.

EXAMPLES.

1. Find $\int x^2 \log x dx$.

Assume $u = \log x$; then

$$dv = x^2dx, \quad du = \frac{dx}{x}, \quad \text{and} \quad v = \int x^2dx = \frac{x^3}{3}.$$

Substituting in (A), we have

$$\int x^2 \log x dx = \frac{x^3}{3} \log x - \int \frac{x^2 dx}{3} = \frac{x^3 \log x}{3} - \frac{x^3}{9} + C.$$

2. Find $\int \sin^{-1} x\, dx$.

Assume $u = \sin^{-1} x$; then $dv = dx$, $v = x$, and $du = \dfrac{dx}{\sqrt{1-x^2}}$.

Substituting in (A), we have

$$\int \sin^{-1} x\, dx = x \sin^{-1} x - \int \frac{x\, dx}{\sqrt{1-x^2}} = x \sin^{-1} x + (1-x^2)^{\frac{1}{2}} + C.$$

Find the following:

3. $\int \tan^{-1} x\, dx$. $\qquad x \tan^{-1} x - \log(1 + x^2)^{\frac{1}{2}} + C.$

4. $\int x \cos x\, dx$. $\qquad x \sin x + \cos x + C.$

5. $\int x e^{ax} dx$. $\qquad e^{ax}\left(\dfrac{x}{a} - \dfrac{1}{a^2}\right) + C.$

Make $\quad dv = e^{ax} dx; \qquad \therefore v = \dfrac{e^{ax}}{a}, \quad u = x.$

Sometimes two or more applications of the formula are required, as in the next example.

6. $\int x^2 e^{ax} dx$. $\qquad e^{ax}\left[\dfrac{x^2}{a} - \dfrac{2x}{a^2} + \dfrac{2}{a^3}\right] + C.$

Make $\quad dv = e^{ax} dx; \quad \therefore v = \dfrac{e^{ax}}{a}, \quad u = x^2, \quad du = 2x\, dx.$

$$\therefore \qquad \int x^2 e^{ax} dx = \frac{e^{ax} x^2}{a} - 2\int \frac{e^{ax} x\, dx}{a}.$$

Now apply the formula to the last term, as in Ex. 5, and we obtain the entire integral.

7. $\int \frac{x^3 dx}{\sqrt{a^2 - x^2}}$. $\qquad - x^2\sqrt{a^2 - x^2} - \frac{2}{3}(a^2 - x^2)^{\frac{3}{2}} + C.$

Make $dv = \frac{x\,dx}{\sqrt{a^2 - x^2}}$; $\therefore\ v = -\sqrt{a^2 - x^2}$, and $u = x^2$.

$$\therefore \qquad \int \frac{x^3 dx}{\sqrt{a^2 - x^2}} = - x^2\sqrt{a^2 - x^2} + \int 2x(a^2 - x^2)^{\frac{1}{2}} dx.$$

In a similar manner we may integrate any binomial differential, or by continued application of the formula reduce it to a simpler form, but by the method of Art. 211 the result may in general be obtained with less labor.

8. $\int x^3 \cos x\, dx$.

Make $\quad dv = \cos x\, dx$; then $v = \sin x$, $u = x^3$, $du = 3x^2 dx$.

$$\therefore \qquad \int x^3 \cos x\, dx = x^3 \sin x - \int 3x^2 \sin x\, dx.$$

Again, make $dv = -\sin x\, dx$; then $v = \cos x$, $u = 3x^2$, $du = 6\,x dx$.

$$\therefore \qquad -\int 3x^2 \sin x\, dx = 3x^2 \cos x - \int 6x \cos x\, dx.$$

Again, make $dv = -\cos x\, dx$, then $v = -\sin x$, $u = 6x$, $du = 6dx$.

$$\therefore\ -\int 6x \cos x\, dx = -6x \sin x + \int 6 \sin x\, dx (= -6 \cos x).$$

$$\therefore\ \int x^3 \cos x\, dx = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C.$$

9. $\int \log x\, dx$. $\qquad x(\log x - 1) + C.$

10. $\int x^3 \log x\, dx$. $\qquad \frac{x^4}{4} \log x - \frac{x^4}{16} + C.$

11. $\int x^2 \log^2 x\, dx$. $\qquad \frac{1}{3}x^3(\log^2 x - \frac{2}{3} \log x + \frac{2}{9}) + C.$

12. $\int e^x x^4 dx.$ $\qquad e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) + C.$

13. $\int x a^x dx.$ $\qquad \frac{a^x}{\log a}\left(x - \frac{1}{\log a}\right) + C.$

14. $\int x^2 e^{-x} dx.$ $\qquad -e^{-x}(x^2 + 2x + 2) + C.$

15. $\int x^3 e^{ax} dx.$ $\qquad \left(x^3 - \frac{3x^2}{a} + \frac{6x}{a^2} - \frac{6}{a^3}\right)\frac{e^{ax}}{a} + C.$

16. $\int x^2 \sin^{-1} x\, dx.$ $\qquad \frac{x^3}{3}\sin^{-1} x + \frac{x^2+2}{9}\sqrt{1-x^2} + C.$

17. $\int \frac{\log x\, dx}{(x+1)^2}.$ $\qquad \frac{x}{x+1}\log x - \log(x+1) + C.$

18. $\int x^3 (\log x)^2 dx.$ $\qquad \frac{x^4}{4}[(\log x)^2 - \frac{1}{2}\log x + \frac{1}{8}] + C.$

19. $\int (a^2 - x^2)^{-\frac{1}{2}} x^2 dx.$ $\qquad -\frac{x}{2}(a^2 - x^2)^{\frac{1}{2}} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$

20. $\int (a^2 + x^2)^{-\frac{1}{2}} x^2 dx.$ $\qquad \frac{x}{2}(a^2+x^2)^{\frac{1}{2}} - \frac{a^2}{2}\log\left(x + \sqrt{x^2+a^2}\right) + C.$

REDUCTION FORMULAS.

198. Reduction formulas are formulas by which the integral of a differential may be made to depend on the integral of a similar, but simpler, differential.

199. To find the reduction formula for $\int x^p (\log x)^n dx$, where n is a positive integer.

Assume $\qquad dv = x^p dx \qquad$ and $\qquad u = (\log x)^n;$

then $\qquad v = \frac{x^{p+1}}{p+1} \qquad$ and $\quad du = n(\log x)^{n-1}\frac{dx}{x}.$

Substituting in (A), Art. 197, we have

$$\int x^p(\log x)^n dx = \frac{x^{p+1}(\log x)^n}{p+1} - \frac{n}{p+1}\int x^p(\log x)^{n-1}dx, \quad (1)$$

in which the proposed integral depends upon another of the same form, but having the exponent of $\log x$ less by one. By successive applications of this formula the exponent of $\log x$ is reduced to zero, and the proposed integral is made to depend upon the known form $\int x^p dx$.

Cor. I. If the given integral were $\int X(\log x)^n dx$, where X is any algebraic function of x, we should have

$$\int X(\log x)^n dx = X_1(\log x)^n - n\int \frac{X_1}{x}(\log x)^{n-1}\, dx,$$

where $X_1 = \int X dx$.

EXAMPLES.

1. $\int x^3 \log^3 x\, dx.$

Here $p = 3, p + 1 = 4, n = 3$ and $n - 1 = 2$. Substituting in (1), we have

$$\int x^3 \log^3 x\, dx = \tfrac{1}{4}x^4 \log^3 x - \tfrac{3}{4}\int x^3 (\log x)^2 dx.$$

By applying the formula to the last term, etc., we obtain

$$\int x^3 \log^3 x = \frac{x^4}{4}\left[\log^3 x - \frac{3}{4}\log^2 x + \frac{3\,.\,2}{4^2}\log x - \frac{3\,.\,2\,.\,1}{4^3}\right] + C.$$

2. $\int x^5 (\log x)^2\, dx.$ $\qquad \dfrac{x^6}{6}\left[(\log x)^2 - \tfrac{1}{3}\log x + \tfrac{1}{18}\right] + C.$

3. $\int \dfrac{\log x\, dx}{(1 + x)^2}.$ $\qquad \dfrac{x}{1+x}\log x - \log(1 + x) + C.$

$$X = \frac{1}{(1 + x)^2}, \quad \therefore X_1 = -\frac{1}{1 + x}.$$

200. To find the reduction formula for $\int a^x x^n dx$, where n is a positive integer.

Let $\quad dv = a^x dx \quad$ and $\quad u = x^n$;

then $\quad v = \dfrac{a^x}{\log a} \quad$ and $\quad du = nx^{n-1}dx.$

$\therefore$ Art. 197, $\quad \displaystyle\int a^x x^n dx = \frac{a^x x^n}{\log a} - \frac{n}{\log a}\int a^x x^{n-1} dx. \quad . \quad . \quad (1)$

EXAMPLES.

1. $\displaystyle\int a^x x^3 dx. \qquad \frac{a^x}{\log a}\left[x^3 - \frac{3x^2}{\log a} + \frac{6x}{\log^2 a} - \frac{6}{\log^3 a}\right] + C.$

Here $n = 3$, $\quad \displaystyle\therefore \int a^x x^3 dx = \frac{a^x x^3}{\log a} - \frac{3}{\log a}\int a^x x^2 dx.$

By further applications of (1) we obtain the desired result.

2. $\displaystyle\int a^x x^4 dx.$

$$\frac{a^x}{\log a}\left[x^4 - \frac{4x^3}{\log a} + \frac{12x^2}{\log^2 a} - \frac{24x}{\log^3 a} + \frac{24}{\log^4 a}\right] + C.$$

3. $\displaystyle\int e^{ax} x^3 dx. \qquad \frac{e^{ax}}{a}\left[x^3 - \frac{3x^2}{a} + \frac{6x}{a^2} - \frac{6}{a^3}\right] + C.$

201. To find the reduction formula for $\int x^n \cos ax\, dx$ and $\int x^n \sin ax\, dx$, where n is a positive integer.

Make $\quad u = x^n \quad$ and $\quad dv = \cos ax\, dx$;

then $\quad du = nx^{n-1}dx \quad$ and $\quad v = \dfrac{\sin ax}{a}.$

$$\therefore \quad \int x^n \cos ax\, dx = \frac{1}{a}x^n \sin ax - \frac{n}{a}\int x^{n-1} \sin ax\, dx.$$

Similarly, we find

$$\int x^n \sin ax\,dx = -\frac{1}{a}x^n \cos ax + \frac{n}{a}\int x^{n-1} \cos ax\,dx.$$

Hence, in either case, the integral can be made to depend on the known form $\int \cos ax\,dx$ or $\int \sin ax\,dx$.

EXAMPLES.

1. $\int x^3 \cos x\,dx.$ $\quad x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + C.$

2. $\int x^4 \sin x\,dx.$

$$-x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x - 24x \sin x - 24 \cos x + C.$$

202. To find the reduction formula for $X \sin^{-1} x\,dx$, $X \tan^{-1} x\,dx$, etc., where X is an algebraic function of x.

Make $\quad u = \sin^{-1} x \quad$ and $\quad dv = Xdx;$

then $\quad du = \frac{dx}{\sqrt{1-x^2}} \quad$ and $\quad v = \int Xdx = X_1$ (say).

$$\therefore \quad \int X \sin^{-1} x\,dx = X_1 \sin^{-1} x - \int \frac{X_1 dx}{\sqrt{1-x^2}}.$$

EXAMPLES.

1. $\int x^2 \tan^{-1} x\,dx.$ $\quad \frac{x^3 \tan^{-1} x}{3} - \frac{x^2}{6} + \frac{\log(1+x^2)}{6} + C.$

Here $\quad X = x^2 \quad$ and $\quad X_1 = \frac{x^3}{3}.$

2. $\int \frac{x^2 \tan^{-1} x\,dx}{1+x^2}.$ $\quad x \tan^{-1} x - \frac{1}{2}(\tan^{-1} x)^2 - \frac{1}{2}\log(1+x^2) + C.$

3. $\int x^2 \sec^{-1} x\,dx.$ $\quad \frac{1}{3}x^3 \sec^{-1} x - \frac{1}{6}(x^2-1)^{\frac{1}{2}}x - \frac{1}{6}\log(x+\sqrt{x^2-1}) + C.$

Reduction formulas for binomial differentials are deduced in Art. 215.

203. To integrate $\dfrac{dx}{a + b \cos x}$.

$$\frac{dx}{a + b \cos x} = \frac{dx}{a\left(\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}\right) + b\left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right)}$$

$$= \frac{dx}{(a + b) \cos^2 \frac{x}{2} + (a - b) \sin^2 \frac{x}{2}}$$

$$= \frac{\sec^2 \frac{x}{2} dx}{(a + b) + (a - b) \tan^2 \frac{x}{2}}.$$

$$\therefore \int \frac{dx}{a + b \cos x} = 2 \int \frac{d\left(\tan \frac{x}{2}\right)}{(a + b) + (a - b) \tan^2 \frac{x}{2}},$$

which is readily reduced to the form

$$\int \frac{dv}{r^2 + v^2} \quad \text{or} \quad \int \frac{dv}{r^2 - v^2}, \quad \text{according as } a > \text{ or } < b.$$

Similarly, $\displaystyle\int \frac{dx}{a + b \sin x}$ can be found.

204. To integrate $\dfrac{dx}{\sin x}$ and $\dfrac{dx}{\cos x}$.

$$\int \frac{dx}{\sin x} = \int \frac{\frac{1}{2} dx}{\sin \frac{1}{2}x \cos \frac{1}{2}x} = \int \frac{\sec^2 \frac{1}{2}x \, \frac{1}{2}dx}{\tan \frac{1}{2}x};$$

$$\therefore \qquad \int \frac{dx}{\sin x} = \log \tan \tfrac{1}{2}x = \log \sqrt{\frac{1 - \cos x}{1 + \cos x}}.$$

Again, $$\int \frac{dx}{\cos x} = \int \frac{dx}{\sin\left(\frac{\pi}{2} - x\right)}$$

$$= - \log \tan \left(\frac{\pi}{4} - \frac{x}{2}\right) + C.$$

APPROXIMATE INTEGRATION.

205. The number of differentials which can be integrated xactly is comparatively very small, yet the approximate value of the integral of any differential may be found when the differential can be developed into a convergent infinite series each of whose terms is integrable. This is the last resort in separating a differential into its integrable parts.

EXAMPLES.

Find the approximate integral of the following:

1. $dy = \dfrac{dx}{a+x}$. $\qquad y = \dfrac{x}{a} - \dfrac{x^2}{2a^2} + \dfrac{x^3}{3a^3} - \dfrac{x^4}{4a^4} + \text{etc.}$

Expanding $\dfrac{1}{a+x}$ by division, we have

$$\frac{1}{a+x} = \frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \frac{x^3}{a^4} + \text{etc.};$$

$$\therefore \qquad y = \int \left(\frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \frac{x^3}{a^4} + \text{etc.}\right) dx.$$

2. $dy = \dfrac{dx}{1+x^2}$. $\qquad y = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \text{etc.}$

By division, $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \text{etc.}$

3. $dy = \dfrac{dx}{\sqrt{1+x^2}}$. $\qquad y = x - \dfrac{x^3}{2\,.\,3} + \dfrac{3x^5}{2\,.\,4\,.\,5} - \dfrac{3\,.\,5x^7}{2\,.\,4\,.\,6\,.\,7} + \text{etc.}$

By the Binomial Theorem,

$$\frac{1}{\sqrt{1+x^2}} = 1 - \frac{x^2}{2} + \frac{3x^4}{2\,.\,4} - \text{etc.}$$

4. $dy = x^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}} dx$. $\qquad y = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{7}x^{\frac{7}{2}} - \frac{1}{44}x^{\frac{11}{2}} - \text{etc.}$

Develop $(1-x^2)^{\frac{1}{2}}$, multiply by $x^{\frac{1}{2}} dx$, etc.

5. $dy = x^3(\cos x)dx$. $\qquad y = \frac{x^4}{4} - \frac{x^6}{12} + \frac{x^8}{192} - \text{etc.}$

6. $dy = x^6 \sin^{-1} x\, dx$. $\qquad y = x^7\left(\frac{1}{7} + \frac{x^2}{54} + \frac{3x^4}{440} + \text{etc.}\right)$.

206. Development of Functions by Exact and Approximate Integration. Two or three examples will suffice to illustrate the process.

EXAMPLES.

1. $\int \frac{dx}{a + x} = \log (a + x)$, exactly (omitting C);

$$\int \frac{dx}{a + x} = \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \text{etc., approximately.}$$

$$\therefore \log (a + x) = \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \text{etc.}$$

2. $\int \frac{dx}{\sqrt{1 + x^2}} = \log (x + \sqrt{1 + x^2})$, exactly;

$$\int \frac{dx}{\sqrt{1 + x^2}} = x - \frac{x^3}{6} + \frac{3x^5}{40} - \text{etc., approximately.}$$

$$\therefore \log (x + \sqrt{1 + x^2}) = x - \frac{x^3}{6} + \frac{3x^5}{40} - \text{etc.}$$

3. $\int \frac{dx}{1 + x^2} = \tan^{-1} x$, exactly;

$$\int \frac{dx}{1 + x^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{etc., approximately.}$$

$$\therefore \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{etc.}$$

CHAPTER IX.

INTEGRATION—(*Continued*).

INDEPENDENT INTEGRATION.

207. Increments Deduced from Differentials. We have seen that the increment of a function is the sum of the differential and the acceleration; hence, when the former is known, we can find the differential by simply removing the acceleration. Taylor's formula enables us to reverse this operation in many cases, and find the increment when the differential is known.

Let $$u = f(x).$$

Increasing x by h, we have, by Taylor's formula,

$$u + \Delta u = f(x+h)$$

$$= f(x) + f'(x)h + f''(x)\frac{h^2}{2} \ldots f^n(x)\frac{h^n}{\lfloor n} \ldots$$

$$\therefore \quad \Delta u = f(x+h) - f(x)$$

$$= f'(x)h + f''(x)\frac{h^2}{2} + \ldots f^n(x)\frac{h^n}{\lfloor n} \ldots, \qquad \text{(A)}$$

in which $du = f'(x)h$.

Therefore, when the differential of a function of x is known, the increment may be found by taking the successive derivatives of the differential coefficient, and substituting them in (A).

When $f^n(x) = 0$, and each of the subsequent derivatives of $f(x) = 0$, the series will be finite and express the exact value of Δu; otherwise the series will be infinite, and, if convergent, will give the approximate value of Δu.

EXAMPLES.

1. If $du = (x^2 - 5x + 6)dx$, what is the value of Δu?

Here $f'(x) = x^2 - 5x + 6$. Differentiating this, we obtain $f''(x) = 2x - 5$, $f'''(x) = 2$, $f^{iv}(x) = 0$. Substituting these values in (A), we have

$$\Delta u = (x^2 - 5x + 6)h + (2x - 5)\frac{h^2}{2} + \frac{h^3}{3}.$$

2. If $du = (x^2 - 3x - 10)h$, what is the value of Δu?

$$\Delta u = (x^2 - 3x - 10)h + (2x - 3)\frac{h^2}{2} + \frac{h^3}{3}.$$

3. If $du = (x^3 - 7x^2 + 12x)dx$, what is the value of au?

$$au = (3x^2 - 14x + 12)\frac{h^2}{2} + (3x - 7)\frac{h^3}{3} + \frac{h^4}{4}.$$

4. Find au when $du = \sin x\, dx$.

$$au = \cos x\frac{h^2}{2} - \sin x\frac{h^3}{6} - \cos x\frac{h^4}{24} + \text{etc.}$$

5. If $du = (\sqrt{1 + x})dx$, by how much will u be increased when x is increased by h?

$$\Delta u = (\sqrt{1 + x})h + (1 + x)^{-\frac{1}{2}}\frac{h^2}{4} - (1 + x)^{-\frac{3}{2}}\frac{h^3}{24} + \text{etc.}$$

6. Find au when $du = \log x\, dx$.

$$au = \frac{h^2}{2x} - \frac{h^3}{6x^2} + \frac{h^4}{12x^3} - \frac{h^5}{20x^4} + \text{etc.}$$

7. A function is increasing at the rate of $4x^3dx$; find its succeeding increment. $\Delta u = 4x^3h + 6x^2h^2 + 4xh^3 + h^4$.

8. At the end of t seconds the velocity of a body is

$$\frac{ds}{dt} = (3t^2 - 2t) \text{ ft. per second;}$$

find the distance it will travel the following second, dt being the unit of time. $\Delta s = (3t^2 - 2t)dt + (3t - 1)dt^2 + dt^3$.

9. The rate of acceleration of the velocity of a body is

$$\frac{dv}{dt} = (3t + 4) \text{ ft. per second;}$$

find the increment (1) of the velocity (v), and (2) of the distance (s) for the following second.

$$\Delta v = (3t + 4)dt + \tfrac{3}{2}dt^2. \quad v = \int(3t + 4)dt = \tfrac{3}{2}t^2 + 4t + C.$$

$$\Delta s = (\tfrac{3}{2}t^2 + 4t + C)dt + (\tfrac{3}{2}t + 2)dt^2 + \tfrac{1}{2}dt^3.$$

208. Increments as Definite Integrals. In Fig. 5, where $u =$ area of $OBPA$, $\Delta u =$ the area of $BCP'P$, which is evidently the integral of du between the limits x and $x + h$. In general

$$\int_x^{x+h} f'(x)dx = f(x + h) - f(x).$$

For, since $\quad df(x) = f'(x)dx, \quad \int f'(x)dx = f(x) + C.$

$$\therefore \qquad \int_x^{x+h} f'(x)dx = \Big[f(x) + C\Big]_x^{x+h} = f(x + h) - f(x).$$

Therefore (A) may be written

$$\int_x^{x+h} f'(x)dx = f'(x)h + f''(x)\frac{h^2}{\underline{|2}} + f'''(x)\frac{h^3}{\underline{|3}} \dots f^n(x)\frac{h^n}{\underline{|n}} \dots \text{(B)}$$

By this formula we can obtain exactly, or in the form of an infinite series, the definite integral of any function of a single variable, and the operation does not involve the reversing of any of the formulas for differentiating. But, in general, this method is much inferior to that of dependent integration, since by the latter many differentials can be integrated in finite terms which by the former could be expressed only in the form of an infinite series. However, it forms an important part of the theory of differentials and integrals, and is often useful as a method of approximation.

More convenient formulas for practical purposes will be derived from (A), but before doing so let us apply (B) to the following illustrative examples.

1. Find the area $BCP'P$, the equation of APP' being

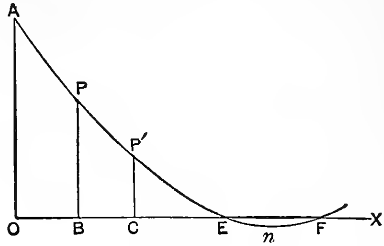

FIG. 43.

$y = x^2 - 7x + 12$, where $x = OB$, $y = BP$, and $BC = h$ or dx.

Let $u =$ area of $OBPA$, then $du = ydx$.

$$\therefore \quad y = f'(x) = x^2 - 7x + 12, \quad f''(x) = 2x - 7,$$
$$f'''(x) = 2, \quad f^{\text{iv}}(x) = 0.$$

Substituting in (B), we have

$$\int_x^{x+h} (y)dx = (x^2 - 7x + 12)h + (2x - 7)\frac{h^2}{2} + \frac{h^3}{3} = \text{area of } BCP'P.$$

COR. I. To find the area of OEA we make $x = 0$ and $h = OE = 3$, and obtain $13\frac{1}{2}$. To find the area of EnF, we make $x = OE = 3$ and $h = EF = 1$, and get $-\frac{1}{6}$.

Let the student solve the following in a similar manner.

2. The equation of a curve is $y = x^3 - 6x^2 + 11x - 6$; find the areas of the two sections enclosed by the curve and the axis of x. $\frac{1}{4}$; $-\frac{1}{4}$.

209. A More Convenient Series. In (A), by making $x = 0$, then making $h = x$, and writing $\int_0^x f'(x)dx$ for $f(x)$, we have

$$\int_0^x f'(x)dx = f(0) + f'(0)x + f''(0)\frac{x^2}{\underline{|2}} + \ldots f^n(0)\frac{x^n}{\underline{|n}} \ldots \quad \text{(C)}$$

This formula may be obtained by developing $f'(x)$ by Maclaurin's formula, multiplying by dx, and integrating each

term separately, but as we are now exemplifying the method of independent integration, we will apply (C) directly to one or two examples.

1. Find $\int_0^x (3x^2 - 14x + 5)dx$.

Here
$$f'(x) = 3x^2 - 14x + 5, \quad \therefore \quad f'(0) = 5;$$
$$f''(x) = 6x - 14, \quad \therefore \quad f''(0) = -14;$$
$$f'''(x) = 6, \quad \therefore \quad f'''(0) = 6.$$

Substituting in (C), we have

$$\int_0^x (3x^2 - 14x + 5)dx = x^3 - 7x^2 + 5x.$$

2. $\int_0^x (x^3 - 6x^2 + 7)dx$. $\qquad \frac{1}{4}x^4 - 2x^3 + 7x$.

3. $\int_0^x (3x^4 - 2x^3 + x^2)dx$. $\qquad \frac{3}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3$.

210. Bernouilli's Series. In formula (B), by making $h = -x$, observing that $\int_0^x f'(x)dx = -\int_x^0 f'(x)dx$, we have

$$\int_0^x f'(x)dx = f'(x)x - f''(x)\frac{x^2}{2} + \ldots - (-1)^n f^n(x)\frac{x^n}{\underline{|n}} \ldots \text{(D)}$$

This formula, called Bernouilli's Series, like formulas (B) and (C), shows the possibility of expressing the integral of every function of a single variable, in terms of that variable, since the successive derivatives $f''(x)$, $f'''(x)$, etc., can always be deduced from $f'(x)$. Hence, in all cases where the series are finite or infinite and convergent the integral may be obtained exactly or approximately.

In finding $\int f'(x)dx$ by (B), (C), or (D) the limits of the difference between the approximate value found and true value may be determined as in A_6 of the Appendix.

INTEGRATION BY INDETERMINATE COEFFICIENTS.

211. The process of integrating binomial and trigonometric differentials by successive reductions is generally very tedious, and it is our purpose now to present a method which is generally less laborious, and which is also applicable to many other classes of differentials.

Let $u^r v dx$ be the differential to be integrated, where u and v are functions of x, and let us assume

$$\int u^r v dx = u^{r+1} f(x) + k \int u^s v_1 dx, \quad . \quad . \quad . \quad . \qquad \text{(E)}$$

in which it is required to find the function $f(x)$ and the constant k.

In examples where $\int u^r v dx$ can be expressed under the form of $u^{r+1} f(x)$, we shall find $k = 0$; and when this is not the case, $\int u^r v dx$ will be determined in terms of $\int u^s v_1 dx$, which we can generally make of a more elementary character by assigning suitable values to s and v_1.

Another advantage of expressing the required integral under the form of (E) arises from the fact that $u^{r+1} f(x)$ often vanishes for the desired limits of integration, in which case the definite integral depends on the last term only.

Differentiating (E) and dividing by $u^r dx$, we have

$$v = (r+1)\frac{du}{dx} f(x) + u f'(x) + k v_1 u^{s-r}. \quad . \quad . \quad . \qquad \text{(F)}$$

The simplest and easiest method of solving this equation for $f(x)$ and k is by indeterminate coefficients, as illustrated in the following examples.

212. Case I. **When $k = 0$;—Independent Integration.**

EXAMPLES.

1. Find $\int (1 + x^2)^{-\frac{1}{2}} x^5 dx$.

Comparing this with (E) we have $u = 1 + x^2$, $r = -\frac{1}{2}$, $\frac{du}{dx} = 2x$, and $v = x^5$. Substituting in (F), we have

$$x^5 = xf(x) + (1 + x^2)f'(x) + kv_1(1 + x^2)^{s+\frac{1}{2}}. \quad . \quad . \quad (1)$$

The first member of this equation being of the fifth degree the second must be also; hence, $f(x)$ must be of the fourth degree, and since u involves only the second power of x, we may assume

$$f(x) = Ax^4 + Bx^2 + C, \quad \therefore f'(x) = 4Ax^3 + 2Bx.$$

Substituting in (1), and arranging in reference to x, we have

$$x^5 = 5Ax^5 + (3B + 4A)x^3 + (C + 2B)x + kv_1(1 + x^2)^{s+\frac{1}{2}}.$$

$$\therefore \qquad 1 = 5A, \quad 3B = -4A, \quad C = -2B, \quad k = 0,$$

or $$A = \tfrac{1}{5}, \qquad B = -\tfrac{4}{15}, \qquad C = \tfrac{8}{15}.$$

These values determine $f(x)$, which substituted in (E) gives

$$\int (1 + x^2)^{-\frac{1}{2}} x^5 dx = (1 + x^2)^{\frac{1}{2}}[\tfrac{1}{5}x^4 - \tfrac{4}{15}x^2 + \tfrac{8}{15}x] + C.$$

2. Find $\int x^{-6}(1 + x^2)^{\frac{1}{2}} dx$. $\qquad -(1 + x^2)^{\frac{3}{2}}\left(\frac{1}{5x^5} - \frac{2}{15x^3}\right) + C.$

Here $u = 1 + x^2$, $r = \frac{1}{2}$, $\frac{du}{dx} = 2x$, and $v = x^{-6}$; $\therefore$ (F) gives

$$x^{-6} = 3xf(x) + (1 + x^2)f'(x) + kv_1(1 + x^2)^{s-\frac{1}{2}}. \quad . \quad . \quad (1)$$

In order that the two members may be of the same degree we assume

$$f(x) = Ax^{-5} + Bx^{-3} + Cx^{-1}; \quad \therefore f'(x) = -5Ax^{-6} - 3Bx^{-4} - Cx^{-2}.$$

Substituting in (1), and arranging in reference to x, we find $A = -\frac{1}{5}$, $B = \frac{2}{15}$, $C = 0$, $k = 0$; this determines $f(x)$; which substituted in (E) gives the desired result.

A careful inspection of the previous examples suggests the following rule for determining the form of $f(x)$ for binomial differentials of the form $(a + bx^n)^p x^m dx$, v being x^m:

I. *When m is positive the highest exponent of x in $f(x)$ will be $m - n + 1$.*

II. *When m is negative the algebraically lowest exponent of x in $f(x)$ will be $- m + 1$.*

III. *The remaining exponents decrease or increase algebraically by n.*

The rule is also applicable when u is a polynomial in which n is the highest exponent of x, provided that the exponents of x in $f(x)$ increase or decrease by the least difference between the exponents of x in u.

When r is a fraction and u a polynomial of a higher degree than the second, the differential cannot ordinarily be integrated; or, more accurately, its integral cannot ordinarily be finitely expressed in terms of the functions with which we are familiar. The exceptional or integrable cases are, in general, where u, v, and r are such that it is possible for $f(x)$ to have as many coefficients as there will be independent equations between the coefficients in equation (F), and where $k\int v_1 u^s dx$ is 0 or one of the integrable forms. In a differential of any given form the conditions of integrability may often be determined by the present method.

3. $\displaystyle\int \frac{dx}{x^4\sqrt{x^2 + 6x + 15}}$. $\qquad \displaystyle\frac{\sqrt{x^2 + 6x + 15}}{-45}\left(\frac{1}{x^3} - \frac{1}{2x^2} + \frac{1}{6x}\right) + C.$

Here $u = x^2 + 6x + 15$, $r = -\frac{1}{2}$, and $v = x^{-4}$; hence the lowest exponent of x in $f(x)$ will be $-4 + 1 = -3$, and the others will increase by 1, giving $fx = Ax^{-3} + Bx^{-2} + Cx^{-1} + D$.

The process can also be applied to many differentials in which v is a polynomial, as in the next example.

4. $\displaystyle\int \frac{3x^2 + 5x + 5}{\sqrt{x^2 + 2x + 3}} dx$. $\qquad \displaystyle\sqrt{x^2 + 2x + 3}\left(\frac{3x + 1}{2}\right) + C.$

Here $u = x^2 + 2x + 3$, $r = -\frac{1}{2}$, $v = 3x^2 + 5x + 5$, and $f(x)$ is of the form $Ax + B$.

5. $\displaystyle\int \frac{dx}{(1+x^2)^{\frac{7}{2}}}$. $\qquad \displaystyle\frac{x}{(1+x^2)^{\frac{5}{2}}}\left[\frac{8}{15}x^4 + \frac{4}{3}x^2 + 1\right] + C.$

Here $u = 1 + x^2$, $r = -\frac{7}{2}$, $v = 1$; and making $v_1 = 1$, $s = -\frac{1}{2}$, (F) gives

$$1 = -5xf(x) + (1 + x^2)f'(x) + k(1 + x^2)^3,$$

where $f(x)$ is evidently of the form $Ax^5 + Bx^3 + Cx$.

213. We have seen (Art. 185) that $\sin^m x \cos^n x\, dx$ can be easily reduced to an integrable form when either m or n, or both, are positive odd integers, or when $m + n$ is an even integer and negative. In such cases, m and n being integers, the integration may be effected by the independent method, as in the two following examples; but this method of integrating such differentials is introduced and recommended chiefly for its bearing on the cases in which the above conditions do not exist, and which are usually solved by successive reductions.

6. $\displaystyle\int \sin^5 x \cos^4 x\, dx$. $\qquad \displaystyle -\cos^5 x\left[\frac{\sin^4 x}{9} + \frac{4\sin^2 x}{63} + \frac{8}{315}\right] + C.$

We may make $u = \cos x$, $r = 4$; then $\dfrac{du}{dx} = -\sin x$ and $v = \sin^5 x$. Substituting in (F), we have

$$\sin^5 x = -5 \sin x f(x) + \cos x f'(x) + kv_1 \cos^{s-4} x,$$

where $f(x)$ is evidently of the form $A \sin^4 x + B \sin^2 x + C$, and hence $f'(x) = (4A \sin^3 x + 2B \sin x) \cos x$.

$$\therefore\ \sin^5 x = -5A \sin^5 x - 5B \sin^3 x - 5C \sin x$$
$$+ (4A \sin^3 x + 2B \sin x) \cos^2 x + \text{etc.}$$

Now, substituting $1 - \sin^2 x$ for $\cos^2 x$, reducing, and arranging with respect to $\sin x$, we have

$$(1 + 9A)\sin^5 x + (7B - 4A)\sin^3 x + (5C - 2B)\sin x + kv_1 \cos^{s-4} x = 0.$$

$$\therefore \quad A = -\tfrac{1}{9}, \quad B = -\tfrac{4}{63}, \quad C = -\tfrac{8}{315}, \quad k = 0.$$

7. $\displaystyle\int \frac{dx}{\cos^6 x}$. $\quad\displaystyle\frac{\sin x}{\cos^5 x}\left[\frac{8}{15}\sin^4 x - \frac{4}{3}\sin^2 x + 1\right] + C.$

Make $u = \cos x$, $r = -6$, $\dfrac{du}{dx} = -\sin x$, $v = 1$, $s = 0$, $v_1 = 1$, and we have from (F)

$$1 = 5\sin x f(x) + \cos x f'(x) + k\cos^6 x.$$

Assume $\quad f(x) = A\sin^5 x + B\sin^3 x + C\sin x.$

$$\therefore \quad f'(x) = 5A\sin^4 x\cos x + 3B\sin^2 x\cos x + C\cos x.$$

Substitute, reduce, etc., and we find

$$A = \frac{8}{15}, \quad B = -\frac{4}{3}, \quad C = 1, \quad k = 0.$$

This method of integration can often be applied to other classes of differentials, as in the next two examples.

8. $\displaystyle\int x^4 \log^2 x\, dx.$ $\quad\displaystyle\frac{x^5}{5}\left(\log^2 x - \frac{2}{5}\log x + \frac{2}{25}\right) + C.$

Make $u = x$, $r = 4$, $v = \log^2 x$, and assume

$$f(x) = A\log^2 x + B\log x + C.$$

9. $\displaystyle\int e^{ax}x^3 dx.$ $\quad\displaystyle e^{ax}\left(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right) + C.$

Make $u = e^{ax}$, $r = 0$, $s = 1$; then $v = e^{ax}x^3$, and $\dfrac{du}{dx} = ae^{ax}$; substituting in (F) and dividing by e^{ax}, we have

$$x^3 = af(x) + f'(x) + kv_1,$$

where we evidently have $f(x) = Ax^3 + Bx^2 + Cx + D$.

Find:

10. $\int \frac{x^3\,dx}{\sqrt{1-x^2}}$. $\quad -\sqrt{1-x^2}\left(\frac{1}{3}x^2+\frac{2}{3}\right)+C.$

11. $\int \frac{x^5\,dx}{\sqrt{1-x^2}}$. $\quad -\sqrt{1-x^2}\left(\frac{1}{5}x^4+\frac{1\,.\,4}{3\,.\,5}x^2+\frac{1\,.\,2\,.\,4}{1\,.\,3\,.\,5}\right)+C.$

12. $\int \frac{x^7\,dx}{\sqrt{1-x^2}}$.

$$-\sqrt{1-x^2}\left(\frac{1}{7}x^6+\frac{1\,.\,6}{5\,.\,7}x^4+\frac{1\,.\,4\,.\,6}{3\,.\,5\,.\,7}x^2+\frac{1\,.\,2\,.\,4\,.\,6}{1\,.\,3\,.\,5\,.\,7}\right)+C.$$

13. $\int x^{-2}(1+x^2)^{-\frac{3}{2}}dx.$ $\quad -\frac{1}{\sqrt{1+x^2}}\left(\frac{1}{x}+2x\right)+C.$

Assume $\quad f(x)=\frac{A}{x}+Cx.$

14. $\int \frac{x^5dx}{\sqrt{a+bx^2}}$. $\quad \sqrt{a+bx^2}\left(3x^4-\frac{4ax^2}{b}+\frac{8a^2}{b^2}\right)\frac{1}{15b}+C.$

15. $\int \frac{dx}{(a+bx^2)^{\frac{5}{2}}}$. $\quad \frac{x}{(a+bx^2)^{\frac{3}{2}}}\left(\frac{2bx^2}{3a^2}+\frac{1}{a}\right)+C.$

16. $\int \frac{(5x^3+3x^2+2)dx}{\sqrt{1+x^3}}$. $\quad 2\sqrt{1+x^3}(x+1)+C.$

17. $\int x^5(2+3x^2)^{\frac{1}{2}}dx.$ $\quad \frac{(2+3x^2)^{\frac{3}{2}}}{27}\left(\frac{9}{7}x^4-\frac{24}{35}x^2+\frac{32}{105}\right)+C.$

18. $\int x^{-4}(1-2x^2)^{-\frac{1}{2}}dx.$ $\quad -(1-2x^2)^{\frac{1}{2}}\left(\frac{1+4x^2}{3x^3}\right)+C.$

19. $\int \frac{(x^4-2x+\frac{1}{3})dx}{\sqrt{x^4-4x+1}}$. $\quad (x^4-4x+1)^{\frac{1}{2}}\left(\frac{x}{3}\right)+C.$

20. $\int \frac{(x^3+x^2)dx}{\sqrt{5\frac{3}{5}+4x+x^2}}$. $\quad (5\frac{3}{5}+4x+x^2)^{\frac{1}{2}}\left(\frac{x^2}{3}-\frac{7x}{6}+\frac{49}{15}\right)+C.$

21. $\int \frac{(2x^5-5x^4-6x+7)dx}{(x^5-5x+2)^{\frac{4}{5}}}$. $\quad (x^5-5x+2)^{\frac{1}{5}}(x-5)+C.$

22. $\int \sin^5 x\cos^3 x\,dx.$ $\quad \frac{1}{8}\sin^6 x(\cos^2 x+\frac{1}{3})+C.$

Make $\quad u=\sin x,\quad r=5,\quad v=\cos^3 x.$

23. $\int \frac{\sin^5 x}{\cos^2 x} dx.$ $\qquad -\frac{1}{3\cos x}\left[\sin^4 x + 4\sin^2 x - 8\right] + C.$

24. $\int x^5 \log^3 x\, dx.$ $\qquad \frac{x^6}{6}\left[\log^3 x - \frac{1}{2}\log^2 x + \frac{1}{6}\log x - \frac{1}{36}\right] + C.$

25. $\int e^{ax} x^4 dx.$ $\qquad e^{ax}\left[\frac{x^4}{a} - \frac{4x^3}{a^2} + \frac{4.3x^2}{a^3} - \frac{4.3.2x}{a^4} + \frac{4.3.2.1}{a^5}\right] + C.$

26. $\int \frac{\log^2 x}{x^3} dx.$ $\qquad -\frac{1}{2x^2}\left[\log^2 x + \log x + \frac{1}{2}\right] + C.$

214. Case II. **When k is not $= 0$;—Dependent Integration.**

EXAMPLES.

1. $\int x^4(1 - x^2)^{\frac{1}{2}} dx.$ $\qquad (1-x^2)^{\frac{1}{2}}\left[\frac{x^5}{6} - \frac{x^3}{24} - \frac{x}{16}\right] + \frac{1}{16}\sin^{-1} x + C.$

The differential $x^4(1 - x^2)^{\frac{1}{2}} dx$ has the same binomial factor as $(1 - x^2)^{-\frac{1}{2}} dx$, whose integral is $\sin^{-1} x$; hence, by expressing the former in terms of the latter, the required integral may be obtained in terms of $\int (1 - x^2)^{-\frac{1}{2}} dx$. Thus,

$$x^4(1 - x^2)^{\frac{1}{2}} = x^4(1 - x^2)(1 - x^2)^{-\frac{1}{2}} = (x^4 - x^6)(1 - x^2)^{-\frac{1}{2}}.$$

$\therefore\ u = 1 - x^2,\ r = -\frac{1}{2},\ \frac{du}{dx} = -2x,\ v = x^4 - x^6$, which substituted in (F) gives (making $s = r$ and $v_1 = 1$)

$$x^4 - x^6 = -xf(x) + (1 - x^2)f'(x) + k,$$

where $f(x) = Ax^5 + Bx^3 + Cx$; and proceeding as before, we find $A = \frac{1}{6}$, $B = -\frac{1}{24}$, $C = -\frac{1}{16}$, and $k = \frac{1}{16}$.

2. $\int \frac{dx}{(1 + x^2)^3}.$ $\qquad \frac{1}{(1 + x^2)^2}\left[\frac{3}{8}x^3 + \frac{5}{8}x\right] + \frac{3}{8}\tan^{-1} x + C.$

Here $v = 1 + x^2$, $r = -3$, $\frac{du}{dx} = 2x$, $v = 1$, and in order to

express $\int \frac{dx}{(1+x^2)^3}$ in terms of $\int \frac{dx}{1+x^2} [= \tan^{-1} x]$ we make $v_1 = 1$, $s = -1$, $s - r = 2$, and (F) becomes

$$1 = -4xf(x) + (1+x^2)f'(x) + k(1 + 2x^2 + x^4),$$

where $f(x) = Ax^3 + Bx$.

3. $\int \frac{dx}{x^5\sqrt{1-x^2}}$. $\quad -\sqrt{1-x^2}\left(\frac{1}{4x^4} + \frac{3}{8x^2}\right) + \frac{3}{8} \log\left(1 - \frac{\sqrt{1-x^2}}{x}\right) + C.$

In order to express the integral in terms of $\int \frac{dx}{x\sqrt{1-x^2}}$ $\left[= \log\left(\frac{1-\sqrt{1-x^2}}{x}\right)\right]$ we make $u^{-r} = x\sqrt{1-x^2} = \sqrt{x^2 - x^4}$, $s = -\frac{1}{2}$, then $u = x^2 - x^4$, $r = -\frac{1}{2}$, $\frac{du}{dx} = 2x - 4x^3$, $v = \frac{1}{x^4}$, and (F) becomes

$$x^{-4} = (x - 2x^3)f(x) + (x^2 - x^4)f'(x) + k,$$

where $f(x) = Ax^{-5} + Bx^{-3} + Cx^{-1}$, since x^2 is a factor of u.

4. $\int \frac{x^3 + 8x + 21}{(x^2 - 4x + 9)^2} dx.$

Here.

$u = x^2 - 4x + 9$, $r = -2$, $\frac{du}{dx} = 2x - 4$, and $v = x^3 + 8x + 21$;

hence we have from (F), making $s = -1$,

$$x^3 + 8x + 21 = (4 - 2x)f(x) + (x^2 - 4x + 9)f'(x) \\ + kv_1(x^2 - 4x + 9). \quad . \quad (1)$$

The second member of this equation must be of the third degree; but if we make $v_1 = 1$, the solution will be impossible, let us therefore assume that $k = 1$ and $v_1 = Cx + D$; we may then make $f(x) = Ax + B$ and $f'(x) = A$. Substituting in (1),

we find $A = \frac{3}{2}$, $B = -\frac{21}{2}$, $C = 1$, and $D = \frac{11}{2}$. Hence the required integral is

$$\frac{3(x-7)}{2(x^2-4x+9)} + \int \frac{(x+\frac{11}{2})dx}{x^2-4x+9}$$

$$= \frac{3(x-7)}{2(x^2-4x+9)} + \int \frac{(x-2)dx}{x^2-4x+9} + \int \frac{\frac{15}{2}dx}{x^2-4x+9}$$

$$= \frac{3(x-7)}{2(x^2-4x+9)} + \tfrac{1}{2}\log(x^2-4x+9) + \frac{3\sqrt{5}}{2}\tan^{-1}\left(\frac{x-2}{\sqrt{5}}\right) + C.$$

The solutions of the three following examples illustrate the manner of integrating $\sin^m x \cos^n x\, dx$ when the conditions stated in Art. 213 do not exist.

5. $\displaystyle\int \frac{dx}{\sin^5 x}$. $\qquad \displaystyle\frac{1}{\sin^4 x}[\tfrac{3}{8}\cos^3 x - \tfrac{5}{8}\cos x] + \tfrac{3}{8}\log\tan\tfrac{1}{2}x + C.$

Here we make $u = \sin x$, $r = -5$; then $v=1$ and $\dfrac{du}{dx} = \cos x$. Making $v_1 = 1$ and $s = -1$, (F) gives

$$1 = -4\cos x f(x) + \sin x f'(x) + k\sin^4 x.$$

Making $\sin^4 x = (1-\cos^2 x)^2$, $f(x) = A\cos^3 x + B\cos x$, and proceeding as in examples 6 and 7, Art. 213, we find $A = \frac{3}{8}$, $B = -\frac{5}{8}$, and $k = \frac{3}{8}$.

$$\therefore \int \frac{dx}{\sin^5 x} = \frac{1}{\sin^4 x}(\tfrac{3}{8}\cos^3 x - \tfrac{5}{8}\cos x) + \tfrac{3}{8}\int \frac{dx}{\sin x}. \quad \text{(Art. 204)}$$

This example may also be solved like the following one.

6. $\displaystyle\int \frac{dx}{\cos^7 x}$.

Make $u = \sin x$, $r = 0$, then $\dfrac{du}{dx} = \cos x$, and $v = \cos^{-7} x$.

$$\therefore \quad \cos^{-7} x = \cos x f(x) + \sin x f'(x) + kv_1 \sin^s x, \quad . \quad (1)$$

where $\qquad f(x) = A\cos^{-6}x + B\cos^{-4}x + C\cos^{-2}x,$

and $\qquad f'(x) = (6A\cos^{-7}x + 4B\cos^{-5}x + 2C\cos^{-3}x)\sin x.$

Substituting in (1), making $\sin^2 x = 1 - \cos^2 x$, reducing, etc., we have

$$\frac{6A-1}{\cos^7 x} + \frac{4B-5A}{\cos^5 x} + \frac{2C-3B}{\cos^3 x} - \frac{C}{\cos x} + kv_1 \sin^s x = 0.$$

$\therefore\ A = \frac{1}{6},\ B = \frac{5}{24},\ C = \frac{15}{48}$, and $\left(\text{making } s=0,\ v_1 = \frac{1}{\cos x}\right) k = C.$

$$\therefore \int \frac{dx}{\cos^7 x} = \frac{\sin x}{6\cos^6 x} + \frac{5.1\sin x}{6.4\cos^4 x} + \frac{5.3.1\sin x}{6.4.2\cos^2 x} + \frac{5.3.1}{6.4.2}\int\frac{dx}{\cos x}.$$

To integrate the last term see Art. 204, page 188.

7. $\int \sin^4 x\cos^4 x\,dx.$

$$\sin^3 x\left[-\frac{\cos^5 x}{8} + \frac{\cos^3 x}{16} + \frac{3\cos x}{64}\right] - \frac{3}{128}(\sin x\cos x - x) + C.$$

Make $u = \sin x$, $r = 2$, then $v = \sin^2 x\cos^4 x = \cos^4 x - \cos^6 x$; also make $s = 2$ and $v_1 = 1$, then

$$\cos^4 x - \cos^6 x = 3\cos x f(x) + \sin x f'(x) + k. \quad . \quad . \quad (1)$$

Make $f(x) = A\cos^5 x + B\cos^3 x + C\cos x$, find $f'(x)$, substitute in (1), for $\sin^2 x$ write $1 - \cos^2 x$, etc., and we find $A = -\frac{1}{8}$, $B = \frac{1}{16}$, $C = \frac{3}{64}$, and $k = \frac{3}{64}$.

$$\int \sin^4 x\cos^4 x\,dx = \sin^3 x\left[-\frac{\cos^5 x}{8} + \frac{\cos^3 x}{16} + \frac{3\cos x}{64}\right] + \frac{3}{64}\int \sin^2 x\,dx.$$

To integrate the last term, see ex. 11, Art. 185.

8. $\displaystyle\int_0^a \frac{x^2dx}{\sqrt{a^2-x^2}}$. $\quad \displaystyle\left[-\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a = \tfrac{1}{4}\pi a^2.$

9. $\displaystyle\int_{-a}^a \frac{dx}{x^3\sqrt{a^2-x^2}}$.

$$\left[-\frac{\sqrt{a^2-x^2}}{2a^2x^2}+\frac{1}{2a^3}\log\frac{x}{a+\sqrt{a^2-x^2}}\right]_{-a}^{+a} = -\frac{\log(-1)}{2a^3}.$$

10. $\displaystyle\int\sqrt{a^2+x^2}\,dx$. $\quad \displaystyle\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right)+C.$

11. $\displaystyle\int(a^2+x^2)^{\frac{3}{2}}dx$.

$$\frac{x}{8}(2x^2+5a^2)\sqrt{x^2+a^2}+\frac{3a^4}{8}\log\left(x+\sqrt{x^2+a^2}\right)+C.$$

12. $\displaystyle\int(a^2-x^2)^{\frac{3}{2}}dx$. $\quad \displaystyle\frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2}+\frac{3a^4}{8}\sin^{-1}\frac{x}{a}+C.$

13. $\displaystyle\int_0^{2a}\frac{x^2dx}{\sqrt{2ax-x^2}}$.

$$\left[-\sqrt{2ax-x^2}\left(\frac{x+3a}{2}\right)+\frac{3a^2}{2}\operatorname{vers}^{-1}\frac{x}{a}\right]_0^{2a} = \tfrac{3}{2}\pi a^2.$$

14. $\displaystyle\int_0^{2a} x^2\sqrt{2ax-x^2}\,dx$. $\quad \tfrac{5}{8}\pi a^4.$

15. $\displaystyle\int_0^{2a} x^3\sqrt{2ax-x^2}\,dx$. $\quad \tfrac{7}{8}\pi a^5.$

16. $\displaystyle\int_0^1\frac{x^4dx}{\sqrt{1-x^2}}$. $\quad \displaystyle\frac{1.3}{2.4}\left(\frac{\pi}{2}\right).$

17. $\displaystyle\int_0^1\frac{x^6dx}{\sqrt{1-x^2}}$. $\quad \displaystyle\frac{1.3.5}{2.4.6}\left(\frac{\pi}{2}\right).$

18. $\displaystyle\int\sqrt{3+2x+x^2}\,dx$.

$$\sqrt{3+2x+x^2}\left(\frac{x+1}{2}\right)+\log\left(2x+2+2\sqrt{3+2x+x^2}\right)+C.$$

19. $\displaystyle\int\sqrt{10+3x-x^2}\,dx$.

$$\sqrt{10+3x-x^2}\left(\frac{2x-3}{4}\right)+\frac{49}{8}\sin^{-1}\frac{2x-3}{7}+C.$$

20. $\displaystyle\int_{-2}^{1}\frac{3x^2-2x+5}{\sqrt{2-x-x^2}}\,dx.$ $\quad\frac{81}{8}\pi.$

21. $\displaystyle\int\frac{dx}{(a^2+x^2)^2}.$ $\quad\displaystyle\frac{x}{2a^2(a^2+x^2)}+\frac{1}{2a^3}\tan^{-1}\frac{x}{a}+C.$

22. $\displaystyle\int\frac{x^5-x^4+21}{(x^2+3)^3}dx.$

$$\frac{6x^3+12x^2+22x+27}{4(x^2+3)^2}+\frac{1}{2}\log(x^2+3)$$
$$+\frac{1}{2\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}+C.$$

23. $\displaystyle\int\frac{(3x+2)dx}{(x^2-3x+3)^2}.$

$$\frac{13x-24}{3(x^2-3x+3)}+\frac{26}{3\sqrt{3}}\tan^{-1}\frac{2x-3}{\sqrt{3}}+C.$$

24. $\displaystyle\int\sin^6 x\,dx.$

$$\sin^3 x(\tfrac{1}{6}\cos^3 x-\tfrac{3}{8}\cos x)-\tfrac{5}{16}\sin x\cos x+\tfrac{5}{16}x+C.$$

25. $\displaystyle\int\frac{\sin^4 x}{\sec^2 x}dx.$ $\quad\displaystyle\frac{\cos x}{2}\left(\frac{\sin^5 x}{3}-\frac{\sin^3 x}{12}-\frac{\sin x}{8}\right)+\frac{x}{16}+C.$

$$\sin^4 x\sec^{-2} x=\sin^4 x\cos^2 x=\sin^4 x-\sin^6 x.$$

Hence we may make $u=\cos x$, $r=0$, and $v=\sin^4 x-\sin^6 x$.

26. $\displaystyle\int\frac{\sin^2 x\,dx}{\cos^5 x}.$ $\quad\displaystyle\frac{\sin x}{4\cos^4 x}-\frac{\sin x}{8\cos^2 x}-\tfrac{1}{8}\log(\sec x+\tan x)+C.$

$$\frac{\sin^2 x}{\cos^5 x}=\frac{1-\cos^2 x}{\cos^5 x}=\frac{1}{\cos^5 x}-\frac{1}{\cos^3 x}.$$

Hence we may make $u=\sin x$, $r=0$, and $v=\cos^{-5}x-\cos^{-3}x$.

27. $\displaystyle\int\cos^4 x\,\mathrm{cosec}^3 x\,dx.$ $\quad\displaystyle-\frac{\cos x}{2\sin^2 x}-\cos x-\tfrac{3}{2}\log\tan\frac{x}{2}+C.$

28. $\displaystyle\int\sin^6 x\sec^4 x\,dx.$ $\quad\displaystyle\frac{\sin^5 x}{3\cos^3 x}-\frac{5\sin^3 x}{3\cos x}+\tfrac{5}{2}[x-\sin x\cos x]+C.$

29. $\displaystyle\int\frac{\sec^3 x}{\sin^4 x}dx.$ $\quad\displaystyle-\frac{1}{\cos^2 x}\left(\frac{1}{3\sin^3 x}+\frac{5}{3\sin x}-\frac{5}{2}\sin x\right)$

$$+\tfrac{5}{2}\log(\sec x+\tan x)+C.$$

215. Reduction Formulas for Binomial Differentials.

These may be easily obtained by the method of indeterminate coefficients.

I. Required $\int (a + bx^n)^p x^m dx$ in terms of

$$\int (a + bx^n)^p x^{m-n} dx.$$

Make $u = a + bx^n$, $r = p$, $\dfrac{du}{dx} = nbx^{n-1}$, $v = x^m$, $s = r$, and $v_1 = x^{m-n}$.

Substituting in (F), we have

$$x^m = (p+1)nbx^{n-1}f(x) + (a + bx^n)f'(x) + kx^{m-n}, \qquad (1)$$

where $f(x) = Ax^{m-n+1}$, and $f'(x) = (m - n + 1)Ax^{m-n}$.

Substituting these values in (1), arranging with reference to x, we find

$$A = \frac{1}{b(np + m + 1)}, \quad \text{and} \quad k = -\frac{a(m - n + 1)}{b(np + m + 1)}.$$

$$\therefore \int (a + bx^n)^p x^m dx = \frac{x^{m-n+1}(a + bx^n)^{p+1}}{b(np + m + 1)}$$
$$- \frac{a(m - n + 1)}{b(np + m + 1)} \int (a + bx^n)^p x^{m-n} dx. \qquad (A)$$

By a repetition of this formula m may be diminished by any integral multiple of n.

II. Required $\int (a + bx^n)^p x^m dx$ in terms of

$$\int (a + bx^n)^{p-1} x^m dx.$$

Make $\quad u = a + bx^n, \quad r = p - 1, \quad s = p - 1,$

$$v = (a + bx^n)x^m, \quad \text{and} \quad v_1' = x^m.$$

Substituting in (F) and proceeding as before, we get

$$\int (a + bx^n)^p x^m dx = \frac{x^{m+1}(a + bx^n)^p}{np + m + 1}$$
$$+ \frac{anp}{np + m + 1}\int (a + bx^n)^{p-1} x^m dx. \quad \text{(B)}$$

Each application of this formula diminishes the exponent of $a + bx^n$ by unity.

When m or p is negative, we need formulas for increasing instead of diminishing them; hence the following:

III. Required $\int (a + bx^n)^p x^m dx$ in terms of

$$\int (a + bx^n)^p x^{m+n} dx.$$

Solving (A) for $\int (a + bx^n)^p x^{m-n} dx$, and substituting $m + n$ for m, we get

$$\int (a + bx^n)^p x^m dx = \frac{x^{m+1}(a + bx^n)^{p+1}}{a(m + 1)}$$
$$- \frac{b(np + n + m + 1)}{a(m + 1)}\int (a + bx^n)^p x^{m+n} dx. \quad \text{(C)}$$

IV. Required $\int (a + bx^n)^p x^m dx$ in terms of

$$\int (a + bx^n)^{p+1} x^m dx.$$

Solving (B) for $\int (a + bx^n)^{p-1} x^m dx$, and substituting $p + 1$ for p, we find

$$\int (a + bx^n)^p x^m dx = - \frac{x^{m+1}(a + bx^n)^{p+1}}{an(p + 1)}$$
$$+ \frac{np + n + m + 1}{an(p + 1)}\int (a + bx^n)^{p+1} x^m dx. \quad \text{(D)}$$

216. The **approximate integral** of many differentials may be conveniently obtained by the method of indeterminate coefficients. The following important example will serve to illustrate the process.

Integrate the **Elliptic Differential**

$$ds = (a^2 - e^2x^2)^{\frac{1}{2}} \frac{dx}{\sqrt{a^2 - x^2}}.$$

Comparing this with (E), we may make $u = a^2 - x^2$, whence $r = -\frac{1}{2}$, $\frac{du}{dx} = -2x$, and $v = (a^2 - e^2x^2)^{\frac{1}{2}}$, which, when developed by the Binomial Theorem, gives

$$v = a - \frac{e^2}{2a}x^2 - \frac{e^4}{8a^3}x^4 - \frac{e^6}{16a^5}x^6 \dots$$

Substituting in (F), Art. 211, making $s = -\frac{1}{2}$, $v_1 = 1$, we have

$$a - \frac{e^2x^2}{2a} - \frac{e^4x^4}{8a^3} - \frac{e^6x^6}{16a^5} \dots = -xf(x) + (a^2 - x^2)f'(x) + k,$$

where $f(x)$ is evidently of the form $fx = \dots Ax^5 + Bx^3 + Cx$.

Proceeding as in Art. 212, we find

$$A = \frac{e^6}{96a^5}, \quad B = \frac{5}{4}Aa^2 + \frac{e^4}{32a^3}, \quad C = \frac{3}{2}Ba^2 + \frac{e^2}{4a}, \quad k = a - Ca^2.$$

$$\therefore \Big[s \Big]_0^x = (a^2 - x^2)^{\frac{1}{2}} \Big[\dots \frac{e^6}{96}\left(\frac{x}{a}\right)^5 + \left(\frac{5e^6}{384} + \frac{e^4}{32}\right)\left(\frac{x}{a}\right)^3$$

$$+ \left(\frac{5e^6}{256} + \frac{3e^4}{64} + \frac{e^2}{4}\right)\left(\frac{x}{a}\right)\Big] + a\left(1 - \frac{e^2}{4} - \frac{3e^4}{64} - \frac{5e^6}{256} \dots\right) \sin^{-1}\frac{x}{a}.$$

CHAPTER X.

INTEGRATION AS A SUMMATION OF ELEMENTS.

ELEMENTS OF FUNCTIONS.

217. Hitherto nothing has been said about the magnitude of differentials. Whether they are large or small does not affect the principles which have been deduced; hence we may regard them as small as we please. They are variables whose limits are zero.

218. In the present chapter increments are called and treated as **Elements.*** Thus Δy or $\Delta f(x)$ $(= m_1 h + m_2 h^2$, Art. 24) is an element of the function $y = f(x)$. For convenience the element $\Delta f(x)$ will often be represented by E_x, and the differential dy or $f'(x)dx$ by D_x, which may be called, respectively, the xth element and the xth differential of the function $f(x)$. Since D_x varies as dx and approaches E_x indefinitely as dx approaches 0, D_x is called the *differential value* of E_x with respect to dx.

The expression $\sum_{x_1}^{x_2}[E_x]$ represents the sum of all the elements like E_x, or the sum of the successive values of E_x, between the x-limits x_1 and x_2. That is, supposing the increment of x to be always h,

$$\sum_{x_1}^{x_2}[E_x] = E_{x_1} + E_{x_1+h} + E_{x_1+2h} + \ldots E_{x_2-h\,(\text{or } x_1+(n-1)h)}.$$

219. A Definite Integral Regarded as a Sum. The practical importance of integration consists chiefly in regarding it

* Because *sum* and *element* are correlative terms.

as the summation of a certain series. For example, in seeking the area of a curve, we conceive it divided into an indefinite number of suitable elementary areas, of which we seek to determine the sum by a process of integration. The solution of this fundamental problem is effected by the following formula and its corollaries.

Suppose that in any function of x, as $f(x)$, we change x from x_1 to x_2 by giving to x successive increments. The whole change in the value of $f(x)$, viz., $f(x_2) - f(x_1)$, must be the sum of the partial changes produced by the increments given to x. That is,

$$f(x_2) - f(x_1) = \Sigma_{x_1}^{x_2}[\Delta f(x)],$$

or (Art. 208) $$\int_{x_1}^{x_2} D_x = \Sigma_{x_1}^{x_2}[E_x]. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A)}$$

Formula (A) is not only an expression of the simple fact that the whole is equal to the sum of its parts or elements, but it signifies that the *integral* of the *differential value of an element*, between certain limits, is the *sum* of the *successive values of that element*, between the same limits.

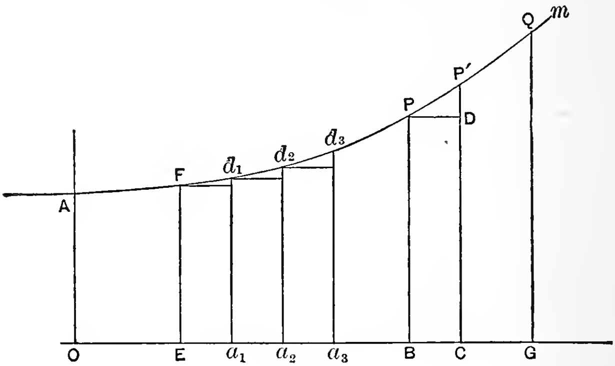

FIG. 43.

As an illustration of (A) let us consider the area of $EGQF$, where $OE = x_1$ and $OG = x_2$. Suppose $y = f'(x)$ to be the equation of $AF'Q$, where $x = OB$ and $y = BP$.

Let $u =$ the area of $OBPA$, and let $BC (= h)$ be an element of x, then $BCDP = du = f'(x)dx = D_x$, and $BCP'P = D_x + au = E_x$.

(*a*) Evidently, $$EGQF = \int_{x_1}^{x_2} D_x, \text{ Art. 208.} \quad . \quad . \quad . \quad (1)$$

(*b*) Divide EG $(= x_2 - x_1)$ into n parts, each equal to h, and draw the ordinates a_1d_1, a_2d_2, a_3d_3, etc., then

$$EGQF = Ed_1 + a_1d_2 + a_2d_3 +, \quad . \quad . \quad . \quad . \quad (2)$$

in which Ed_1, a_1d_2, a_2d_3, etc., are the successive values of E_x, as x increases by h from x_1 to x_2. That is, $Ed_1 = E_{x_1}$, $a_1d_2 = E_{x_1+h}$, $a_2d_3 = E_{x_1+2h}$, etc. Hence (2) may be written

$$EGQF = \Sigma_{x_1}^{x_2}[E_x]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now, equating (1) and (3), and we obtain formula (A).

In further illustration of formula (A), let us show that the signification which it expresses is true of $\int_{x_1}^{x_2} 3x^2 dx$.

(*a*) $$\int_{x_1}^{x_2} 3x^2 dx = [x^3 + C]_{x_1}^{x_2} = x_2^3 - x_1^3. \quad . \quad . \quad . \quad (4)$$

(*b*) Since $f'(x) = 3x^2$, $E_x = 3x^2h + 3xh^2 + h^3$, Art. 207,

$$\begin{aligned}
\therefore \quad E_{x_1} &= 3x_1^2h + 3x_1h^2 + h^3; \\
E_{x_1+h} &= 3x_1^2h + 9x_1h^2 + 7h^3; \\
E_{x_1+2h} &= 3x_1^2h + 15x_1h^2 + 19h^3; \\
&. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
E_{x_1+(n-1)h} &= 3x_1^2h + 3x_1(2n-1)h^2 + (3n^2 - 3n + 1)h^3.
\end{aligned}$$

Taking the sum, remembering that $x_1 + nh = x_2$, and, by Algebra, $3 + 9 + 15 + \ldots 3(2n-1) = 3n^2$, and $1 + 7 + 19 + \ldots (3n^2 - 3n + 1) = n^3$, we have

$$\begin{aligned}
\Sigma_{x_1}^{x_2}[E_x] &= 3nhx_1^2 + 3(nh)^2x_1 + (nh)^3 \\
&= (x_1 + nh)^3 - x_1^3 = x_2^3 - x_1^3. \quad . \quad . \quad . \quad (5)
\end{aligned}$$

Comparing (4) and (5), we see that the results of the operations indicated in (A), when applied to $\int_{x_1}^{x_2} 3x^2 dx$, are the same.

220. Formula (A) is also true when $x_2 - x_1$ ($= EG$) is not divided into equal parts.

Let us suppose $x_2 - x_1$ to be divided into the following equal or unequal positive parts:

$$a - x_1, \quad b - a, \quad c - b, \ldots l - k, \quad x_2 - l,$$

the sum of which is evidently $x_2 - x_1$; then we have identically

$$\int_{x_1}^{x_2} y\,dx = \int_{x_1}^{a} y\,dx + \int_{a}^{b} y\,dx + \int_{b}^{c} y\,dx + \ldots \int_{k}^{x_2} y\,dx, \qquad (1)$$

in which $a - x_1$, $b - a$, $c - b$, etc., may be considered the successive values of Δx and $\int_{x_1}^{a} y\,dx$, $\int_{a}^{b} y dx$, etc., the corresponding successive values of E_x. Hence (1) is the general signification of (A), which the student may easily illustrate with a figure.

221. A Definite Integral Regarded as the Limit of a Sum. In Fig. 43 let us suppose n to increase and h to decrease, nh being always equal to EG. Since the limit of $E_x \div D_x$, as h approaches 0, is unity, the sum of all the rectangles like D_x approaches indefinitely the constant sum of all the elements like E_x. Therefore

$$\lim_{h \doteq 0} \sum_{x_1}^{x_2} \Big[D_x \Big] = \sum_{x_1}^{x_2} \Big[E_x \Big].$$

Substituting in (A), Art. 219, we have

$$\int_{x_1}^{x_2} D_x = \lim_{h \doteq 0} \sum_{x_1}^{x_2} \Big[D_x \Big]. \quad . \quad . \quad . \quad . \quad . \quad (B)$$

That is, the definite integral $\int_{x_1}^{x_2} D_x$ is equal to the limit of the sum of all the successive values of D_x, as x increases by h from x_1 to x_2.

For example, let us find the value of $\int_{x_1}^{x_2} 3x^2 dx$.

Since $\quad D_x = 3x^2h,\quad$ we have

$$D_{x_1} = 3x_1{}^2h,$$

$$D_{x_1+h} = 3x_1{}^2h + 6x_1h^2 + 3h^3,$$

$$D_{x_1+2h} = 3x_1{}^2h + 12x_1h^2 + 12h^3,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$D_{x_1+(n-1)h} = 3x_1{}^2h + 6(n-1)x_1h^2 + 3(n-1)^2h^3.$$

Taking the sum, remembering that

$$0 + 6 + 12 + \ldots 6(n-1) = 3(n^2 - n),$$

and $0 + 3 + 12 + 27 + \ldots 3(n-1)^2 = \dfrac{(n-1)(n)(2n-1)}{2}$,

we have

$$\Sigma_{x_1}^{x_2}[D_x] = 3nhx_1{}^2 + 3(n^2-n)x_1h^2 + \frac{(n-1)(n)(2n-1)}{2}h^3$$

$$= 3(nh)x_1{}^2 + 3(nh)^2x_1 - 3(nh)x_1h + (nh)^3 - \tfrac{3}{2}(nh)^2h + \left(\frac{nh}{2}\right)h^2.$$

Now, making $nh = x_2 - x_1$, and then passing to the limit by making $h = 0$, we have

$$\underset{\Delta x \doteq 0}{\text{limit}}\ \Sigma_{x_1}^{x_2}[D_x] = 3(x_2 - x_1)x_1{}^2 + 3(x_2 - x_1)^2x_1 + (x_2 - x_1)^3$$

$$= x_2{}^3 - x_1{}^3,$$

which is evidently equal to $\int_{x_1}^{x_2} 3x^2 dx$.

222. It is important to observe that, whether an integral be regarded as a *sum*, or the *limit* of a sum, *integrating* is equivalent to two distinct operations:

(*a*) If a *sum*, as in Art. 219, integrating $f'(x)dx$ is equivalent to (1) increasing the differential $f'(x)dx$ by the acceleration au to obtain the element E_x, and (2) finding the sum of the successive values of E_x.

(*b*) If the *limit* of a sum, as in Art. 221, integrating $f'(x)dx$ is equivalent to (1) finding the sum of the successive values of $f'(x)h$, and (2) taking the limit of the sum, as h approaches 0.

In case (*a*) all the quantities involved are finite; but in case (*b*) the limit of each *part* is 0, and the limit of the number of parts is ∞. Both methods have their advantages, and hence both will be employed, more or less, in the applications which follow.

Just here the student may profitably read Art. 238, which offers a simple illustration of the significations and practical importance of formulas (A) and (B).

APPLICATIONS TO GEOMETRY.*

223. Lengths of Curves.—I. Rectangular Co-ordinates.

To find the length (s) of the arc APQ between the limits $OB = x_1$ and $OG = x_2$. (Fig. 44.)

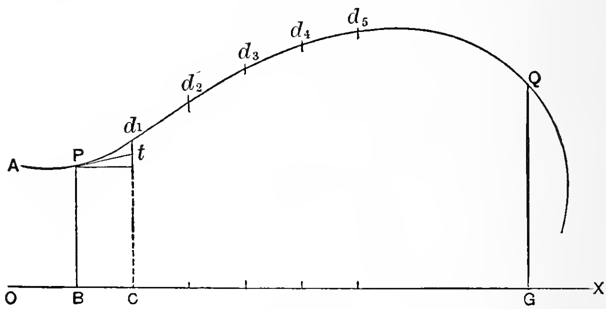

Fig. 44.

Here $E_x = Pd_1$ and $D_x = Pt = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, h$, Art. 33, and

* The previous applications of Calculus to Geometry, Arts. 62, 64, 65, 66, were limited to the most elementary rules for integration; in this chapter it is our purpose to extend these applications by the more advanced methods of integration with which the student is now familiar, and in doing so to impress upon him the important principle of integration as a summation.

$$Pd_1 + d_1d_2 + d_2d_3 + \text{etc.} = \Sigma_{x_1}^{x_2}[E_x] \text{ or } \underset{h \doteq 0}{\text{limit}} \Sigma_{x_1}^{x_2}[D_x] = \int_{x_1}^{x_2} D_x.$$

$$\therefore \quad s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{or} \quad \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy. \qquad \text{(C)}$$

EXAMPLES.

1. Find the length of the arc of the curve $y = \frac{x^3}{6} + \frac{1}{2x}$ between the limits $x_1 = 1$ and $x_2 = 2$.

Here $\quad \frac{dy}{dx} = \frac{x^4 - 1}{2x^2}; \quad \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = \frac{1 + x^4}{2x^2} = \frac{ds}{dx}.$

$$\therefore \quad s = \int_1^2 \left(\frac{1 + x^4}{2x^2}\right) dx = \int_1^2 \frac{dx}{2x^2} + \int_1^2 \frac{x^2 dx}{2}$$

$$= \left[-\frac{1}{2x} + \frac{x^3}{6}\right]_1^2 = \tfrac{17}{12}.$$

2. Rectify the parabola $y^2 = 4ax$, using the formula

$$\frac{ds}{dy} = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}.$$

$$\frac{dx}{dy} = \frac{y}{2a}; \quad \therefore \frac{ds}{dy} = \sqrt{\frac{y^2}{4a^2} + 1} = \frac{1}{2a} \sqrt{y^2 + 4a^2}.$$

$$\therefore \quad \Big[s\Big]_0^y = \frac{1}{2a} \int_0^y (4a^2 + y^2)^{\frac{1}{2}} dy$$

$$= \frac{y\sqrt{4a^2 + y^2}}{4a} + a \log\left(\frac{y + \sqrt{4a^2 + y^2}}{2a}\right). \quad \text{Art. 214, Ex. 10.}$$

3. Rectify the curve $y = \log (x + \sqrt{x^2 - 1})$.

$$s = \sqrt{x^2 - 1} + C.$$

4. Rectify the ellipse $y^2 = (1 - e^2)(a^2 - x^2)$.

$$\frac{dy}{dx} = -(1 - e^2)\frac{x}{y} = -\frac{x\sqrt{1 - e^2}}{\sqrt{a^2 - x^2}}.$$

To find the length of a quadrant, we must integrate between the limits 0 and a; hence

$$s = \int_0^a \sqrt{1 + \frac{x^2(1-e^2)}{a^2 - x^2}}\,dx = \int_0^a \sqrt{a^2 - e^2x^2}\frac{dx}{\sqrt{a^2 - x^2}}$$

$$= a\frac{\pi}{2}\left(1 - \frac{1}{4}e^2 - \frac{1 \,.\, 3}{2^2 \,.\, 4^2}e^4 - \frac{1 \,.\, 3^2 \,.\, 5}{2^2 \,.\, 4^2 \,.\, 6^2}e^6 - \text{etc.}\right). \quad \text{Art. 216.}$$

5. Rectify the cycloid $x = r \operatorname{vers}^{-1}\frac{y}{r} - \sqrt{2ry - y^2}$.

Here $$dx = \frac{y\,dy}{\sqrt{2ry - y^2}};$$

$$\therefore \quad ds = \sqrt{dy^2 + \frac{y^2dy^2}{2ry - y^2}} = dy\sqrt{\frac{2r}{2r - y}}.$$

$$\therefore \quad s = \int (2r)^{\frac{1}{2}}(2r - y)^{-\frac{1}{2}}dy = -2\sqrt{2r(2r - y)} + C.$$

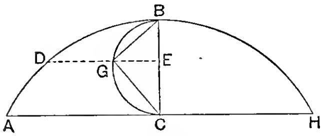

FIG. 45.

If we estimate the arc from the point B where $y = 2r$, we shall have, when $s = 0$, $y = 2r$; hence,

$$0 = 0 + C, \quad \therefore C = 0, \text{ and}$$

$$s = -2\sqrt{2r(2r - y)}.$$

Since $BC = 2r$ and $BE = 2r - y$, $BG = \sqrt{2r(2r - y)}$.

$$\therefore \qquad BD = s = 2BG;$$

or *the arc of a cycloid, estimated from the vertex of the axis, is equal to twice the corresponding chord of the generating circle; hence the entire arc BDA is equal to twice the diameter BC, and the entire curve ADBH is equal to four times the diameter of the generating circle.*

6. Rectify $y = \frac{x^4}{16} + \frac{1}{2x^2}$. $\qquad s = \frac{x^4}{16} - \frac{1}{2x^2} + C.$

NOTE.—The value of C depends on the point from which s is measured. Thus, if s is estimated from $x = 1$, then $s = 0$ when $x = 1$, and we have $0 = \frac{1}{16} - \frac{1}{2} + C$; that is, $C = \frac{7}{16}$.

7. Rectify $y = \dfrac{x^{n+1}}{4(n+1)} + \dfrac{1}{(n-1)x^{n-1}}$.

$$s = y - \frac{2}{(n-1)x^{n-1}} + C.$$

8. Rectify $y = \dfrac{x^m}{2m} - \dfrac{x^{2-m}}{2(2-m)}$. $\quad s = y + \dfrac{x^{2-m}}{2-m} + C.$

9. Rectify $y = \frac{1}{2}\log(x^2 + 3x + 2)$.

$$s = x + \tfrac{1}{2}\log\left(\frac{x+1}{x+2}\right) + C.$$

10. Rectify $y = \frac{1}{8}x^2 - \log x$. $\qquad s = \frac{1}{8}x^2 + \log x + C.$

A curve is said to be *rectifiable* when its length can be expressed in finite terms by aid of the algebraic and elementary transcendental functions.

224. II. Polar Co-ordinates. To find the length (s) of the arc APQ between the limits $\theta_1 = AOP$ or $r_1 = OP$, and $\theta_2 = AOQ$ or $r_2 = OQ$. (Fig. 46.)

Here $E_\theta = Pd_1$ and $D_\theta = Pt = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$, Art. 97; hence

$$Pd_1 + d_1d_2 + d_2d_3 + \text{etc.} = \Sigma_\theta^{\theta_2}[E_\theta]$$

or $$\lim_{\Delta\theta \doteq 0} \Sigma_\theta^{\theta_2}[D_\theta] = \int_\theta^{\theta_2}[D_\theta].$$

$$\therefore\quad s = \int_{\theta_1}^{\theta_2}\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)^{\frac{1}{2}} d\theta, \text{ or } \int_{r}^{r_2}\left(1 + \left(\frac{r\,d\theta}{dr}\right)^2\right)^{\frac{1}{2}} dr. \quad \text{(D)}$$

11. Rectify the spiral of Archimedes, $r = a\theta$.

Here $\dfrac{d\theta}{dr} = \dfrac{1}{a}$; $\quad\therefore\ s = \displaystyle\int_0^r \left(1 + \frac{r^2}{a^2}\right)^{\frac{1}{2}} dr = \frac{1}{a}\int_0^r (a^2 + r^2)^{\frac{1}{2}} dr$

$$= \frac{r(a^2+r^2)^{\frac{1}{2}}}{2a} + \frac{a}{2}\log\frac{r + \sqrt{a^2 + r^2}}{a}. \text{ Art. 214, Ex. 10.}$$

Let this result be compared with the length of the parabola, Ex. 2.

12. Find the length of the logarithmic spiral $\log_a r = \theta$.

Here $$d\theta = \frac{m\,dr}{r}; \quad \therefore \frac{d\theta}{dr} = \frac{m}{r},$$

and $$s = \int_0^r (1 + m^2)^{\frac{1}{2}} dr = r(1 + m^2)^{\frac{1}{2}}.$$

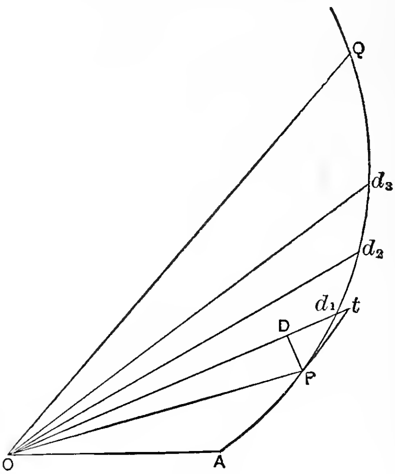

FIG. 46.

13. Find the length, measured from the origin, of the curve

$$y = a \log \frac{a^2 - x^2}{a^2}. \qquad a \log \frac{a + x}{a - x} - x.$$

14. Find the entire length of the arc of the hypocycloid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}. \qquad 6a.$$

15. Find the entire length of the cardioid $r = a(1 - \cos\theta)$.

$8a$.

16. Find the entire length of the curve $r = a \sin^3 \frac{\theta}{3}$. $\quad \frac{3\pi a}{2}$.

17. Find the length, between $x = a$ and $x = b$, of the curve

$$e^y = \frac{e^x + 1}{e^x - 1}. \qquad \log \frac{e^{2b} - 1}{e^{2a} - 1} + a - b.$$

18. Find the length of the tractrix, measured from $(0, a)$, its differential equation being

$$\frac{ds}{dy} = -\frac{a}{y}. \qquad a \log \frac{a}{y}.$$

19. Find the length of the arc, measured from the vertex, of the catenary

$$y = \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right). \qquad \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right).$$

20. Find the length of a quadrant of the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1. \qquad \frac{a^2 + ab + b^2}{a + b}.$$

225. To find the equation of a curve when its length is given.

1. Find the equation of a curve whose length is

$$s = \tfrac{1}{2} \log x + \frac{x^2}{4}.$$

$$\frac{ds}{dx} = \frac{1 + x^2}{2x}; \quad \therefore \sqrt{\left(\frac{ds}{dx}\right)^2 - 1} = \frac{1 - x^2}{2x} = \frac{dy}{dx}.$$

Hence, $dy = \dfrac{1 - x^2}{2x} dx$, and $y = \frac{1}{2} \log x - \frac{1}{4}x^2 + C.$

EXAMPLES.

Find the equations of the curves whose lengths are:

2. $s = -\dfrac{1}{4x^2} + \dfrac{x^4}{8}.$ $\qquad y = s - \dfrac{x^4}{4} + C.$ See Note, p. 219.

3. $s = x + \log\left(\frac{x-1}{x+1}\right).$ $\qquad y = \log(x^2 - 1) + C.$

4. $s = \frac{1}{2}\log(\tan x)$ $\qquad y = \frac{1}{2}\log(\sin 2x) + C.$

5. $s = -\frac{1}{2(n-1)x^{n-1}} + \frac{x^{n+1}}{2(n+1)}.$ $\qquad y = s - \frac{x^{n+1}}{n+1} + C.$

226. Areas of Curves.—I. Rectangular Co-ordinates. To find the area of the surface between a given curve, the axis of x and two ordinates whose abscissas are x_1 and x_2, we have, Art. 219,

$$u = \int_{x_1}^{x_2} y\,dx = \Sigma_{x_1}^{x_2}[E_x]. \quad . \quad . \quad . \quad . \quad . \quad (E)$$

For a definite area between the curve and axis of y, we have

$$u = \int_{y_1}^{y_2} x\,dy = \Sigma_{y_1}^{y^2}[E_y]. \quad . \quad . \quad . \quad . \quad . \quad (F)$$

EXAMPLES.

Find the area of the curve $y = \frac{x^4}{16} + \frac{1}{2x^2}$ between the limits $x_1 = 1$ and $x_2 = 2$.

$$\int_1^2 y\,dx = \int_1^2 \left(\frac{x^4}{16} + \frac{1}{2x^2}\right)dx = \left[\frac{x^5}{80} - \frac{1}{2x}\right]_1^2 = \frac{51}{80}.$$

2. Find the area of the circle $y^2 = a^2 - x^2$.

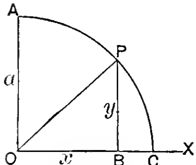

FIG. 47.

$$\text{Area of } OBPA = \int_0^x y\,dx$$

$$= \int_0^x (a^2 - x^2)^{\frac{1}{2}}dx = \frac{x(a^2 - x^2)^{\frac{1}{2}}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

(Ex. 17, Art. 96)

$$= \frac{(OB)(BP)}{2} + \frac{(OP)(\text{arc } AP)}{2} = \text{area } OBP + \text{area } OPA.$$

To find the area of the quadrant OCA we have

$$\int_0^a (a^2 - x^2)^{\frac{1}{2}} dx = \tfrac{1}{4}\pi a^2.$$

The value of π is given in Art. 129.

3. Find the area of an ellipse, $a^2y^2 = a^2b^2 - b^2x^2$.

$$y = \frac{b}{a}\sqrt{a^2 - x^2}; \quad \therefore\ u = \frac{b}{a}\int \sqrt{a^2 - x^2}\, dx.$$

$$\tfrac{1}{4}\text{ (the area)} = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx = \tfrac{1}{4}\pi ab;$$

$\therefore$ $$\text{entire area} = \pi ab.$$

4. Find the area of the hyperbola, $a^2y^2 = b^2x^2 - a^2b^2$.

$$y = \frac{b}{a}\sqrt{x^2 - a^2}; \quad \therefore\ u = \frac{b}{a}\int \sqrt{x^2 - a^2}\, dx,$$

or

$$u = \frac{bx(x^2 - a^2)^{\frac{1}{2}}}{2a} - \frac{ab}{2}\log\left(x + \sqrt{x^2 - a^2}\right) + C.$$

To find C, we know that when $x = a$, $u = 0$; hence

$$0 = -\frac{ab}{2}\log a + C; \quad \therefore\ C = \frac{ab}{2}\log a.$$

Substituting this value of C, and making $\sqrt{x^2 - a^2} = \dfrac{ay}{b}$, we have

$$\text{Area of hyperbola} = \frac{xy}{2} - \frac{ab}{2}\log\left(\frac{x}{a} + \frac{y}{b}\right).$$

5. Find the area of the surface between the arc of the parabola $y^2 = 4ax$ and the axis of y. $\tfrac{1}{3}xy$.

227. It is often convenient and suggestive to regard a definite integral like

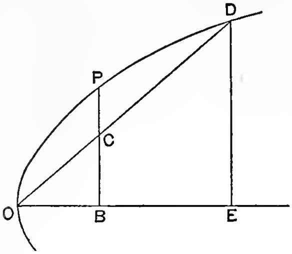

FIG. 48.

$$\int_{x_1}^{x_2} y(dx) = EGQF \qquad \text{(Fig. 43)}$$

as signifying that "the ordinate y (or generatrix PB), moving perpendicularly to the axis of x from $x=x_1$ to $x = x_2$, generates $EGQF$.

Thus, let it be required to find the area of the surface between the parabola $y^2 = 4ax$ and the straight line $y = x$. We at once have

$$OPDC = \int_0^{x_2 = OE} (PC)dx = \int_0^{4a} (\sqrt{4ax} - x)dx = \tfrac{8}{3}a^2.$$

This method can be employed, with equal facility, in finding the volumes of many solids, in which case the generatrix is a surface.

6. Find the area of one branch of the cycloid.

$$dx = \frac{y\,dy}{\sqrt{2ry - y^2}}; \quad \therefore\ u = \int \frac{y^2dy}{\sqrt{2ry - y^2}}.$$

For the area of one branch we have

$$u = 2\int_0^{2r} \frac{y^2dy}{\sqrt{2ry - y^2}} = 3\pi r^2, \qquad \text{(Ex. 13, Art. 214)}$$

which is three times that of the generating circle.

7. Find the area of the curve $xy = a$. $\qquad [u]_c^b = a \log \dfrac{b}{c}.$

8. Find the area of the curve $x^3y - x + 1 = 0$ between the x-limits 1 and 2. $\qquad \tfrac{1}{8}.$

9. Find the area of both loops of $a^4y^2 = a^2b^2x^2 - b^2x^4$. $\qquad \tfrac{4}{3}ab.$

10. Find the area of both loops of the curve $a^6y^2 = a^2x^2 - x^4$.

$\frac{4}{3}$.

11. Prove that the area of the curve $a^{12}y^3 = (a^3 - x^3)x^6$, between the x-limits 0 and a, is the same for all values of a.

228. II. Polar Co-ordinates. In Fig. 46 the area of POD ($= \frac{1}{2}r^2d\theta$, Art. 35) is the differential value of the element POd_1; therefore

$$\int_\theta^{\theta_2} \tfrac{1}{2}r^2d\theta = POd_1 + d_1Od_2 + d_2Od_3 + \ldots d_{n-1}OQ,$$

where $\theta = AOP$ and $\theta_2 = AOQ$.

$$\therefore \qquad u = \tfrac{1}{2}\int_\theta^{\theta_2} r^2d\theta = \text{area } POQ. \quad . \quad . \quad . \quad . \quad . \quad (G)$$

12. Find the area of the spiral of Archimedes, $r = a\theta$.

$$d\theta = \frac{dr}{a}; \quad \therefore u = \frac{1}{2a}\int_0^r r^2dr = \frac{r^3}{6a}.$$

Cor. I. If $a = \dfrac{1}{2\pi}$, as is usual, $u = \frac{1}{3}\pi r^3$.

If $r = 1$, or $\theta = 2\pi$, $u = \frac{1}{3}\pi$, which is the area described by one revolution of the radius vector.

If $r = 2$, or $\theta = 4\pi$, $u = \frac{8}{3}\pi$, which is the area described by two revolutions of the radius vector, which includes the first spire twice; hence the area of the entire spire is $\frac{8}{3}\pi - \frac{1}{3}\pi = \frac{7}{3}\pi$.

13. Find the area of the hyperbolic spiral, $r = \dfrac{a}{\theta}$.

$$du = \frac{a^2d\theta}{2\theta^2}; \quad \therefore \left[u\right]_\theta^{\theta_2} = \left[-\frac{a^2}{2\theta} + C\right]_\theta^{\theta_2} = \frac{a^2}{2\theta} - \frac{a^2}{2\theta_2}.$$

14. Find the area of the logarithmic spiral $\theta = \log_a r$.

$$d\theta = \frac{mdr}{r}; \quad \therefore \text{when } m = 1,$$

$$du = \frac{r\,dr}{2} \quad \text{and} \quad u = \tfrac{1}{4}r^2;$$

that is, the area of the natural logarithmic spiral is equal to one fourth the square described on the radius vector.

15. Find the entire area within the hypocycloid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}. \qquad \tfrac{3}{8}\pi a^2$$

16. Find the area of the surface between the parabola $x^2 = 4ay$ and the witch

$$y = \frac{8a^3}{x^2 + 4a^2}. \qquad (2\pi - \tfrac{4}{3})a^2$$

17. Find the entire area of the cardioid

$$r = a(1 - \cos\theta). \qquad \tfrac{3}{2}\pi a^2.$$

18. Find the area of a loop of the curve

$$x^4 + y^4 = a^2 xy. \qquad \tfrac{1}{8}\pi a^2.$$

19. Find the area of the loop of the curve

$$a^3 y^2 = x^4(b + x). \qquad \frac{32b^{\frac{7}{2}}}{105a^{\frac{3}{2}}}.$$

20. Find the area included between the axes and the curve

$$\left(\frac{x}{a}\right)^{\frac{1}{3}} + \left(\frac{y}{b}\right)^{\frac{1}{3}} = 1. \qquad \frac{ab}{20}.$$

21. Find the area between the curve $x^2y^2 + a^2y^2 = a^2x^2$ and one of its asymptotes. $2a^2$.

22. Find the area of the loop of the curve

$$y^3 + ax^2 - axy = 0. \qquad \tfrac{1}{60}a^2.$$

23. Find the area of the three loops of the curve

$$r = a \sin 3\theta. \quad \text{(See Fig. 38.)} \qquad \tfrac{1}{4}\pi a^2.$$

24. Show that the whole area of $r = a(\sin 2\theta + \cos 2\theta)$ is equal to that of a circle whose radius is a.

229. Areas of Surfaces of Revolution. By Art. 34 the differential value of the xth element of surface is

$$2\pi y\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} dx;$$

therefore

$$S = 2\pi \int_{x_1}^{x_2} y\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} dx = \Sigma_{x_1}^{x_2}[E_x]. \quad . \quad . \quad \text{(H)}$$

EXAMPLES.

1. Find the area of the surface generated by revolving the arc of the curve $y = \frac{x^4}{16} + \frac{1}{2x^2}$ about the axis of x, between $x_1 = 2$ and $x_2 = 4$.

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{ds}{dx} = \frac{x^3}{4} + \frac{1}{x^3}.$$

$$\therefore \qquad S = 2\pi \int_2^4 \left(\frac{x^4}{16} + \frac{1}{2x^2}\right)\left(\frac{x^3}{4} + \frac{1}{x^3}\right) dx.$$

$$= \left[\pi\left[\frac{x^8}{256} + \frac{3x^2}{16} - \frac{1}{4x^4}\right] + C\right]_2^4 = 257\tfrac{271}{1024}\pi.$$

2. Find the area of the surface of a prolate spheroid, the generating curve being the ellipse $a^2y^2 = b^2(a^2 - y^2)$.

$$y = \frac{b}{a}\sqrt{a^2 - x^2}, \quad \text{and} \quad ds = \sqrt{\frac{a^2 - e^2x^2}{a^2 - x^2}}\,dx.$$

$$\therefore \text{ Area} = 2\int_0^a 2\pi y ds = 4\pi\frac{b}{a}\int_0^a (a^2 - e^2x^2)^{\frac{1}{2}} dx$$

$$= 4\pi\frac{b}{a}\left[\frac{x(a^2 - e^2x^2)^{\frac{1}{2}}}{2} + \frac{a^2}{2e}\sin^{-1}\frac{ex}{a}\right]_0^a$$

$$= 2\pi b^2 + \frac{2\pi ab}{e}\sin^{-1} e.$$

3. Find the area of the surface generated by the revolution of the cycloid about its base.

$$\text{Area} = 2\int_0^{2r} 2\pi y ds = 4\pi\sqrt{2r}\int_0^{2r} \frac{y\,dy}{\sqrt{2r - y}}$$

$$= 4\pi\sqrt{2r}\Big[-\tfrac{2}{3}(4r+y)(2r-y)^{\frac{1}{2}}\Big]_0^{2r}$$

$$= \frac{64}{3}\pi r^2.$$

4. Find the surface of the paraboloid between the limits $x = 0$ and $x = a$, the generating curve being $y^2 = 4ax$.

$\frac{1}{3}(\sqrt{8} - 1)8\pi a^2$.

5. Find the surface generated when the cycloid revolves about the tangent at its vertex. $\frac{32}{3}\pi r^2$.

6. Find the surface generated by the revolution about the axis of x of the portion of the curve $y = e^x$, which is on the left of the axis of y. $\pi\big(\sqrt{2} + \log(1 + \sqrt{2})\big)$.

7. Find the entire surface of the oblate spheroid produced by the revolution of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$ about its minor axis.

$$2\pi a^2 + \pi\frac{b^2}{e}\log\frac{1+e}{1-e}.$$

8. Find the surface generated when the cycloid revolves about its axis. $8\pi r^2(\pi - \frac{4}{3})$.

9. Find the surface generated by revolving the arc of the cardioid $r = a(1 - \cos\theta)$ about the initial axis.

$$S = 2\pi\int y\,ds = 2\pi\int r\sin\theta\sqrt{dr^2 + r^2d\theta^2}. \qquad \tfrac{32}{5}\pi a^2.$$

10. Find the surface generated by the revolution of a loop of the lemniscate $r^2 = a^2\sin 2\theta$ about the polar axis. $2\pi a^2$.

230. Volumes of Solids of Revolution. In Art. 32 the volume of the cylinder $(\pi y^2 dx)$ generated by the revolution of the rectangle $BCDP$ is the differential value of the xth element of volume; therefore

$$v = \Sigma_{x_1}^{x_2}[E_x] \quad \text{or} \quad \underset{h \doteq 0}{\text{limit}}\ \Sigma_{x_1}^{x_2}[\pi y^2 h] = \pi\int_{x_1}^{x_2} y^2dx. \qquad \text{(J)}$$

EXAMPLES.

1. Find the volume of a paraboloid, the generating curve being the parabola $y^2 = 4ax$.

$$v = \pi \int_0^x y^2 dx = 4\pi a \int_0^x x\, dx = 2\pi a x^2.$$

When the curve is revolved about the axis of y, we evidently have

$$v = \pi \int x^2 dy.$$

2. Find the volume generated by revolving the surface between the parabola $y = +\sqrt{4ax}$ and the axis of y about that axis.

$$[v]_0^y = \pi \int x^2 dy = \pi \int \frac{y^4}{16a^2} dy = \frac{\pi y^5}{80a^2} = \tfrac{1}{5}\pi x^2 y.$$

That is, the entire volume is one fifth of the volume of the circumscribing cylinder; therefore the volume generated by the surface of the parabola in revolving it about the axis of y is four fifths of that of the circumscribing cylinder.

3. Find the volume of the solid generated by the revolution of the cycloid about its base.

$$dx = \frac{y\, dy}{\sqrt{2ry - y^2}}; \quad \therefore dv = \pi y^2 dx = \frac{\pi y^3 dy}{\sqrt{2ry - y^2}}.$$

To obtain half of the volume, we must integrate between the limits $y = 0$ and $y = 2r$.

$$\therefore \qquad v = 2\pi \int_0^{2r} \frac{y^3 dy}{\sqrt{2ry - y^2}} = 5\pi^2 r^3.$$

That is, $v = \frac{5}{8}$ of the circumscribed cylinder.

4. Find the volume generated by revolving the ellipse AA' about the tangent $X'X$ as an axis. (Fig. 49.)

Let $OA = a$, $OY = b$, $O'B = x$, $BP = y$, and $BP' = y'$;

then $\quad y = \frac{b}{a}(a + \sqrt{a^2 - x^2})$, and $y' = \frac{b}{a}(a - \sqrt{a^2 - x^2})$.

Supposing $P'n = dx$, the volume generated by $P'm$, viz.,

$$\pi(y^2 - y'^2)dx \quad \text{or} \quad 4\pi\frac{b^2}{a}\sqrt{a^2 - x^2}dx,$$

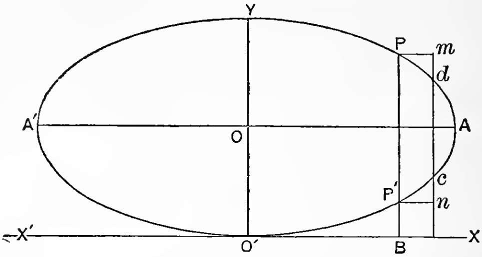

FIG. 49.

is the differential value of the volume of the element generated by $P'cdP$, with respect to dx.

$$\int_{-a}^{+a} 4\pi\frac{b^2}{a}\sqrt{a^2 - x^2}dx = 2\pi^2ab^2, \qquad \text{(Ex. 17, p. 56)}$$

which is the entire volume, being the sum of the volumes generated by revolving all the elements like $P'cdP$ between $x = -a$ and $x = a$, or the limit of the sum of the volumes generated by all the rectangles like $P'nmP$.

5. Find the volume of the closed portion of the solid generated by the revolution of the curve $(y^2 - b^2)^2 = a^3x$ around the axis of y. $\frac{256}{315}\frac{\pi b^9}{a^6}$.

6. Find the volume generated by revolving the curve $(x - 4a)y^2 = ax(x - 3a)$ about the axis of x, between the x-limits 0 and $3a$. $\frac{\pi a^3}{2}(15 - 16 \log 2)$.

7. Find the volume generated by revolving the cycloid round the tangent at the vertex. π^2r^3.

8. Find the volume and surface of the torus generated by revolving the circle $x^2 + (y - b)^2 = a^2$ about the axis of x.

$2\pi^2a^2b$ and $4\pi^2ab$.

9. Find the entire volume and surface generated by revolving the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the axis of X.

$$\frac{32\pi a^3}{105} \text{ and } \frac{12\pi a^2}{5}.$$

10. Find the volume generated by the curve $xy^2 = 4a^2(2a - x)$ revolving about its asymptote. $4\pi^2 a^3$.

11. One branch of the sinusoid $y = b \sin \frac{x}{a}$ is revolved about the axis of x; find the volume generated. $\frac{1}{2}\pi^2 ab^2$.

SUCCESSIVE INTEGRATION.

231. A Double Integral is the indicated result of reversing the operations represented by $\frac{d^2u}{dx\,dy}$.

Thus, if $\frac{d^2u}{dy\,dx} = xy^2$, then $u = \int\int xy^2 dy\,dx$,

which indicates two successive integrations, the first with reference to x, regarding y as constant, and the second with reference to y, regarding x as constant.

$$\text{Thus,} \qquad \int\int xy^2 dy\,dx = \int\left(\frac{x^2}{2} + C\right)y^2 dy$$

$$= \frac{x^2y^3}{6} + \frac{Cy^3}{3} + C_1,$$

where C and C_1 are the constants of integration.

232. Definite Double Integrals. Here both the integrations are between given limits.

$$\text{For example,} \qquad \int_b^c \int_0^a x^3y^2 dx\,dy.$$

This notation indicates that the integrations are to be taken in the following order:

$$\int_b^c\int_0^a x^3y^2dx\,dy = \int_b^c\left(\int_0^a x^3y^2dy\right)dx$$

$$= \int_b^c\left(\frac{a^3x^3}{3}\right)dx = \frac{a^3}{12}(c^4 - b^4).$$

That is, as dy is written last, the y-integration is taken first.

The limits of the first integration are often functions of the second variable.

For example,

$$\int_0^a\int_0^{\sqrt{x}} dx\,dy = \int_0^a(\sqrt{x})dx = \tfrac{2}{3}a^{\frac{3}{2}}.$$

As another example,

$$\int_{\frac{b}{2}}^b\int_0^{\frac{r}{b}} rdr\,d\theta = \int_{\frac{b}{2}}^b\left(\frac{r^2}{b}\right)dr = \tfrac{7}{24}b^2.$$

233. **A Triple Integral** is the indicated operations of three successive integrations, for which the notation is similar to that of double integrals. Thus,

$$\int_0^a\int_0^x\int_0^y x^3y^2z\,dx\,dy\,dz = \int_0^a\left[\int_0^x\left(\int_0^y x^3y^2zdz\right)dy\right]dx.$$

$$= \int_0^a\left[\int_0^x(\tfrac{1}{2}x^3y^4)dy\right]dx = \int_0^a[\tfrac{1}{10}x^8]dx = \tfrac{1}{90}a^9.$$

EXAMPLES.

Find the following:

1. $\int_0^b\int_0^a(x^2+y^2)dx\,dy.$ $\qquad \tfrac{1}{3}ab(a^2+b^2).$

2. $\int_0^a\int_0^{\sqrt{a^2-x^2}}(x^2+y^2)dx\,dy.$ $\qquad \tfrac{1}{8}\pi a^4.$

3. $2a\int_0^a\int_0^{\sqrt{a^2-ax}}\dfrac{dx\,dz}{\sqrt{ax-x^2}}$. $\qquad 4a^2$.

4. $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}\int_0^{\frac{x^2+y^2}{a}} dx\,dy\,dz$. $\qquad \frac{3}{4}\pi a^3$.

AREAS OF SURFACES DETERMINED BY DOUBLE INTEGRATION.

234. Plane Surfaces—(a) Rectangular Co-ordinates. In the formula $u=\int y\,dx$, Art. 226, we may make $y=\int dy$, which gives

$$u=\int\int dx\,dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{J})$$

235. (b) **Polar Co-ordinates.** In the formula $u=\int\frac{r^2}{2}d\theta$,

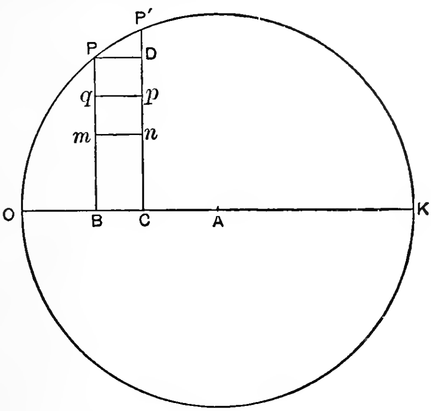

FIG. 50.

Art. 228, we may make $\frac{r^2}{2}=\int r\,dr$, and write

$$u=\int\int r d\theta dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{K})$$

As illustrative examples let us employ (J) and (K) in finding the area of a circle whose radius is a.

Let (x, y) be the co-ordinates of the point m, and $(x + dx, y + dy)$ of the point p, then $mnpq = dxdy$.

Regarding x as constant, we have

$$BCDP = \sum_{y=0}^{y=BP} [mnpq]$$

$$= dx \int_0^{\sqrt{2ax-x^2}} dy = \sqrt{2ax - x^2}\,dx.$$

Again, since $\sqrt{2ax - x^2}\,dx$ is the differential value of BP' with respect to dx, we have

$$\sum_0^{2a} [BP'] = \int_0^{2a} \sqrt{2ax - x^2}\,dx = \tfrac{1}{2}\pi a^2 = \text{area of } OKP.$$

(b) Let (r, θ) be the co-ordinates of m, and $(r + dr, \theta + d\theta)$ of p', then $mn = dr$, $mq = rd\theta$, and $rd\theta dr$ ($=$ area of $mnpq$) is

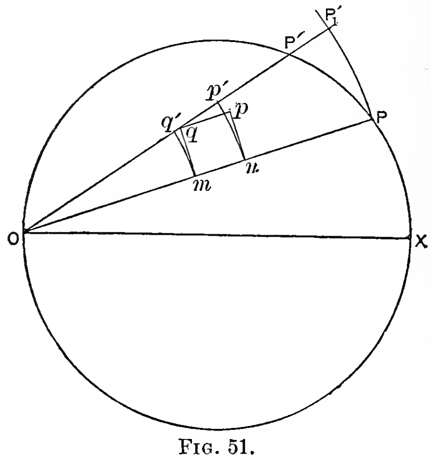

Fig. 51.

the differential value of the element $mnp'q'$ with respect to dr. Therefore, regarding θ as constant, we have

$$\sum_0^{OP} [mnp'q'] = \int^{2a\cos\theta} rd\theta dr = 2a^2 \cos^2 \theta d\theta = OPP_1'.$$

Again, since the area of OPP_1' is the differential value of the element OPP', with respect to $d\theta$, we have

$$\sum_0^{\frac{1}{2}\pi}[OPP'] = \int_0^{\frac{1}{2}\pi} 2a^2 \cos^2 \theta d\theta = \tfrac{1}{2}\pi a^2,$$

which is one half the area of the circle.

EXAMPLES.

1. Find the area (1) of a rectangle by double integration; (2) of a triangle.

2. Find the area between the parabola $y^2 = ax$ and the circle $y^2 = 2ax - x^2$. $2\left(\frac{\pi a^2}{4} - \frac{2a^2}{3}\right)$.

3. Find by double integration the entire area of the cardioid $r = a(1 - \cos \theta)$. $\frac{3\pi a^2}{2}$.

4. In a similar manner find the entire area of the Lemniscate $r^2 = a^2 \cos 2\theta$. a^2.

5. Find the whole area between the curve $xy^2 = 4a^2(2a - x)$ and its asymptote. $4\pi a^2$.

236. Surfaces in General.—To find the area ($= S$) of a surface whose equation is $f(x, y, z) = 0$.

Let (x, y, z) be the co-ordinates of any point P of the surface, and $(x + dx, y + dy, z + dz)$ the co-ordinates of a second point Q very near the first (Fig. 52). Draw planes through P and Q parallel to the planes XZ and YZ. These planes will intercept a curved quadrilateral PQ on the surface; its projection pq, a rectangle, on the plane of XY; and a parallelogram $p'q'$, not shown in the figure, on the tangent plane at P, of which pq is the projection. The area of $p'q' = dS$, since it is the differential value of PQ ($= \Delta S$) with respect to dx and dy.

The projection of $p'q'$ on XY is $dxdy$; similarly the projections of $p'q'$ on XZ and YZ are $dxdz$ and $dydz$; hence, denoting

the angles between the plane of $p'q'$ and XY, XZ and YZ by α, β and γ, respectively, we have

$$\cos \alpha dS = dxdy, \quad \ldots \ldots \quad (1)$$

$$\cos \beta dS = dxdz, \quad \ldots \ldots \quad (2)$$

$$\cos \gamma dS = dydz. \quad \ldots \ldots \quad (3)$$

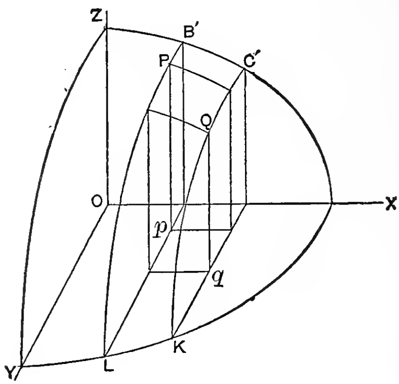

FIG. 52.

Squaring (1), (2), (3), and adding, remembering that

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1,$$

we have

$$(dS)^2 = dx^2dy^2 + dx^2dz^2 + dy^2dz^2;$$

hence,
$$dS = \left(1 + \left(\frac{dz}{dy}\right)^2 + \left(\frac{dz}{dx}\right)^2\right)^{\frac{1}{2}} dxdy.$$

$\therefore$
$$S = \int\int \left(1 + \left(\frac{dz}{dy}\right)^2 + \left(\frac{dz}{dx}\right)^2\right)^{\frac{1}{2}} dxdy. \quad \text{(L)}$$

EXAMPLES.

1. Find the area of one eighth of the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Here $$\frac{dz}{dx} = -\frac{x}{z}, \quad \frac{dz}{dy} = -\frac{y}{z}.$$

$\therefore$ $$1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2} = 1 + \frac{x^2}{z^2} + \frac{y^2}{z^2} = \frac{a^2}{z^2}.$$

Substituting in (L), we have

$$S = \int\int \frac{adxdy}{z} = a\int\int \frac{dxdy}{\sqrt{a^2 - x^2 - y^2}}.$$

Integrating first with reference to y between the limits $y = 0$ and $y = \sqrt{a^2 - x^2}$, we get the differential value of the element $B'C'KL$; and then integrating with reference to x between the limits $x = 0$ and $x = a$, we get the sum of all the elements like $B'C'KL$ between these limits, which sum is the area required.

Hence $$S = a\int_0^a \int_0^{\sqrt{a^2 - x^2}} \frac{dx\,dy}{\sqrt{a^2 - x^2 - y^2}} = \frac{\pi a^2}{2}.$$

2. The two cylinders $x^2 + z^2 = a^2$ and $x^2 + y^2 = a^2$ intersect at right angles; find the surface of the one intercepted by the other. $8a^2$.

Here $z = \sqrt{a^2 - x^2}$, and for one eighth of the required surface the y-limits are 0 and $\sqrt{a^2 - x^2}$, and the x-limits 0 and a.

3. A sphere whose radius is a is cut by a right circular cylinder, the radius of whose base is $\frac{a}{2}$, and one of whose edges passes through the centre of the sphere; find the area of the surface of the sphere intercepted by the cylinder. $2a^2(\pi - 2)$.

Take $x^2 + y^2 + z^2 = a^2$ for the sphere, and $x^2 + y^2 = ax$ for the cylinder, then $z = \sqrt{a^2 - y^2 - x^2}$, and for one fourth of the

required surface the limits of y and x are 0, $\sqrt{ax - x^2}$, and 0, a, respectively.

4. In the preceding example find the surface of the cylinder intercepted by the sphere. (See Ex. 3, Art. 233.) $4a^2$.

5. Find the area of the portion of the surface of the sphere $x^2 + y^2 + z^2 = 2ay$ lying within the paraboloid $y = mx^2 + nz^2$.

$$\frac{2\pi a}{\sqrt{mn}}.$$

237. To find the volume of a solid bounded by a surface whose equation is $f(x, y, z) = 0$.

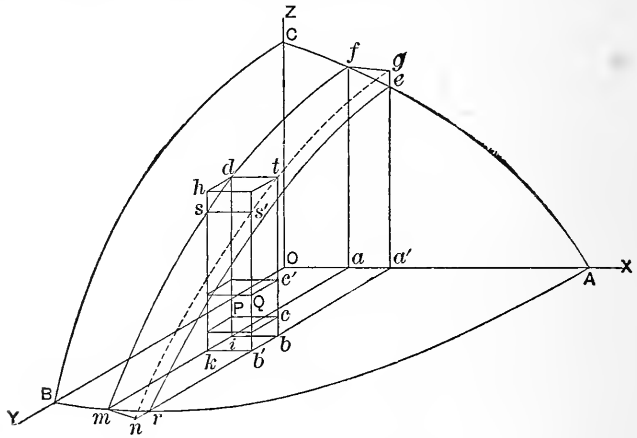

FIG. 53.

VOLUMES OF SOLIDS DETERMINED BY TRIPLE INTEGRATION.

Let $v =$ the indefinite volume expressed by the product of x, y, z; then $v = xyz$, which may be written

$$v = \int\int\int dx\, dy\, dz,$$

which becomes definite when the integrations are taken between certain limits, and we will now give the geometrical interpretation of the formula, step by step.

Let (x, y, z) be the co-ordinates of the point P, and $(x + dx, y + dy, z + dz)$ be the co-ordinates of the point Q; then

$$PQ = dx\,dy\,dz.$$

(*a*) Regarding x and y as constant and integrating between the z-limits 0 and id, we have

$$\Sigma_0^{id}[PQ] = \int_0^{id} dx\,dy\,dz = (id)dx\,dy = bh.$$

(*b*) The volume of bh is the differential value of the element is' with respect to dy; hence

$$\Sigma_0^{am}[is'] = \int_0^{am}(id)dx\,dy = (afm)dx,$$

which is the volume of the cylindrical segment $afm\text{-}a'$.

(*c*) The volume of $afm\text{-}a'$ is the differential value of the element ar with respect to dx; hence

$$\Sigma_0^{OA}[ar] = \int_0^{OA}(afm)dx = \text{volume of } OBC\text{-}A.$$

$$\therefore \qquad \int_0^{OA}\int_0^{am}\int_0^{id} dx\,dy\,dz = \text{volume of } OBC\text{-}A. \quad . \quad . \quad \text{(M)}$$

COR. I. The limits of y and z are found thus: id is the positive result of solving the equation $f(x, y, z) = 0$ for z, am is the positive result of solving $f(x, y, 0) = 0$ for y, and OA is the positive result of solving $f(x, 0, 0) = 0$ for x.

EXAMPLES.

1. Find the volume of one eighth of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The limits of z in this case are 0 and $id = c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}}$;

the limits of y are 0 and $am = b\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}}$; and the limits of x are 0 and a. Therefore the required volume is

$$\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dx\,dy\,dz = \frac{\pi abc}{6}.$$

2. Find the volume of the solid contained between

(*a*) the paraboloid of revolution, $x^2 + y^2 = az$,

(*b*) the cylinder, $x^2 + y^2 = 2ax$,

(*c*) and the plane, $z = 0$.

$\frac{3\pi a^3}{2}$.

The z-limits are 0 and $\frac{x^2 + y^2}{a}$, the y-limits are 0 and $\sqrt{2ax - x^2}$, and the x-limits are 0 and $2a$, for one half of the required volume.

3. Find the volume cut from a sphere whose radius is a by a right circular cylinder whose radius is b, and whose axis passes through the centre of the sphere. $\frac{4\pi}{3}\left(a^3 - (a^2 - b^2)^{\frac{3}{2}}\right)$.

4. Find the entire volume bounded by the surface whose equation is $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = a^{\frac{2}{3}}$. $\frac{4\pi a^3}{35}$.

5. Find the volume of the conoid bounded by the surface $z^2x^2 + a^2y^2 = c^2x^2$, and the planes $x = 0$ and $x = a$. $\frac{1}{2}\pi ac^2$.

APPLICATION TO MECHANICS.

238. Work is said to be done when a body moves through space in opposition to resistance. A horse in drawing a cart or a plough does a certain amount of work, which depends on the resistance and the distance traversed. The force which the horse exerts, and the distance through which he moves, may be regarded as the two elements of the work done. If r lbs. is the constant resistance or force, and x feet the effective distance

through which the body moves, rx units of work will be done. By effective distance is meant the distance measured in the same direction as that in which the force is acting. Thus, when the resistance is constant, the amount of work may be represented by the area of a rectangle whose base is the distance (x) and whose altitude is the resistance (r).

If the resistance or force is a variable dependent on the distance x, it may be represented by $f'(x)$, in which case the amount of work may be found by taking the sum of its elements, thus: In Fig. 43, if the force $f'(x)$ $(= BP)$ act through the small effective distance h $(= BC)$, the work done will be in excess of $f'(x)h$ $(= BCDP)$ only by the acceleration of work (PDP') during that interval. Hence, $f'(x)dx$ is the differential value of the xth element of work. Therefore the quantity of work between the limits $x = x_1$ and $x = x_2$ is, viz.:

$$\Sigma_{x_1}^{x_2}[E_x] \text{ or } \underset{h \doteq 0}{\text{limit}} \Sigma_{x_1}^{x_2}[f'(x)h] = \int_{x_1}^{x_2} f'(x)dx, \quad . \quad \text{(N)}$$

in which the effective distance is $x_2 - x_1$.

COR. I. Effective distance, resistance, and work, and effective distance, force, and energy, bear the same relation to one another as the abscissa, ordinate, and area of a plane curve referred to rectangular co-ordinates, respectively.

EXAMPLE. Let it be required to compute the quantity of work necessary to compress the spiral spring of the common spring-balance to any given degree, say from AB to DB.*

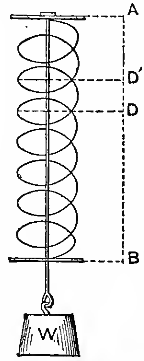

FIG. 54.

Let the resistance $(= f'(x))$ vary directly as the degree of compression, and denote the distance AD' by x; then will

$$f'(x) = mx,$$

where m is the resistance of the spring when the balance

---

* Bartlett's Analytical Mechanics, page 39.

is compressed through the distance unity. Substituting in (N), making $x_1 = 0$ and $x_2 = AD$, we have

$$\text{the work} = \int_0^{x_2} mx\,dx = \left[\frac{mx^2}{2} + C\right]_0^{x_2} = \tfrac{1}{2}mx_2^2.$$

COR. I. If $m = 10$ pounds and $x_2 = 3$ ft., then will

$$\text{the work} = 45 \text{ units of work;}$$

that is, the quantity of work will be equal to that required to raise 45 pounds through a vertical height of one foot.

CENTRE OF GRAVITY.

239. The bodies here considered are supposed to be of uniform density; that is, equal quantities of a body have equal weights.

The **Centre of Gravity** of a body is a point so situated that the force of gravity produces no tendency in the body to rotate about any axis passing through this point.

The **Moment** of any element or particle of a body with reference to any horizontal axis is the product of the magnitude or weight of the element by the horizontal distance of its centre of gravity from the axis, and measures the tendency of the element, under the influence of gravity, to produce rotation about the axis. The moment of the body itself is the sum of the moments of its elements. If the axis of reference passes through the centre of gravity of the body, the moment of the body must be zero, otherwise the moments of the elements would not neutralize one another, and the body would rotate.

240. To find the centre of gravity of a plane area.

In Fig. 43, suppose the plane curve placed in a horizontal position, and let $A =$ the area of $EGQF$, $x = OB$, $y = BP$, $x_1 = OE$, $x_2 = OG$. Also let (x', y') be the centre of gravity of A, and $(x + \alpha, y + \beta)$ be the centre of gravity of the rectangle $BCDP$ $(= yh)$. Evidently the limit of α, as BC or h approaches 0, is 0.

The moment of $BCDP$ with respect to an axis passing through (x', y') and parallel to the axis of y is $(x + \alpha - x')yh$, which is the measure of the tendency of this rectangle $(BCDP)$ to produce rotation about the given axis, and therefore the tendency of all the similar rectangles to produce rotation is $\sum_{x_1}^{x_2}[x + \alpha - x']yh$. The smaller the rectangles the nearer their sum comes to the whole area of the curve, and therefore the tendency of A to rotate about the given axis is

$$\underset{h \text{ or } \alpha \doteq 0}{\text{limit}} \sum_{x_1}^{x_2}(x + \alpha - x')yh = \int_{x_1}^{x_2}(x - x')ydx;$$

but as the axis of reference passes through the centre of gravity of A, this must equal zero.

$$\therefore \quad \int_{x_1}^{x_2}(x - x')ydx = \int_{x_1}^{x_2}xydx - x'\int ydx = 0.$$

$$\text{Whence} \quad x' = \int_{x_1}^{x_2}xydx \div \int_{x_1}^{x_2}ydx = \int_{x_1}^{x_2}xydx \div A. \qquad \text{(P)}$$

In like manner we find

$$y' = \int_{x_1}^{x_2}y^2dx \div A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(Q)}$$

241. To find the centre of gravity of a plane curve.

In Fig. 44, suppose the plane curve $PQ\ (= s)$ placed in a horizontal position; let $x = OB$, $y = BP$, $x_2 = OG$, and $h = BC$; also let (x', y') be the centre of gravity of s, and $(x + \alpha, y + \beta)$ the centre of gravity of the tangent

$$Pt\left(= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}h\right).$$

The limit of α, as h approaches 0, is evidently 0.

The moment of Pt with respect to an axis passing through (x', y') and parallel to the axis of y is $(x+\alpha-x')\sqrt{1+\left(\frac{dy}{dx}\right)^2}h$, and the sum of the moments of all the tangents like Pt is $\Sigma_x^{x_2}[x+\alpha-x']\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}h$. The limit of this sum, as h and α approach 0, is equal to $\int_x^{x_2}(x-x')ds$, which is the moment of s with respect to the given axis, since $s = \underset{h \doteq 0}{\text{limit}} \Sigma_{x_1}^{x_2}[Pt]$.

$$\therefore \qquad \int_x^{x_2}(x-x')ds = \int_x^{x_2}xds - x'\int ds = 0.$$

Whence $$x' = \int_x^{x_2}xds \div s. \quad . \quad . \quad . \quad . \quad . \quad (R)$$

In like manner we find

$$y' = \int_{x_1}^{x_2}yds \div s. \quad . \quad . \quad . \quad . \quad . \quad (S)$$

242. Let the student prove in a similar manner that the formula for finding the centre of gravity of a solid of revolution whose axis is the axis of x is

$$\left.\begin{aligned} x' &= \pi\int_{x_1}^{x_2}xy^2dx \div \text{the volume;} \\ \text{or} \qquad x' &= \int_{x_1}^{x_2}xy^2dx \div \int_{x_1}^{x_2}y^2dx. \end{aligned}\right\} \quad . \quad . \quad . \quad (T)$$

NOTE.—As the centre of gravity must evidently be on the axis of revolution, the formula given above entirely determines it. The same is true of the following formula.

Prove that the formula for finding the centre of gravity of any surface of revolution whose axis is the axis of x is

$$x' = \int_{x_1}^{x_2}xyds \div \int_{x_1}^{x_2}yds. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (U)$$

EXAMPLES.

1. Determine the centre of gravity (G) of an isosceles triangle.

Let OD, the altitude, $= a$, DC, half of the base, $= b$, $OB = x$, $BP = y$; then $y = \frac{bx}{a}$, and, by formula (P),

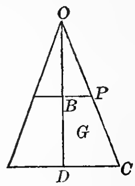

Fig. 55.

$$x' = OG = \frac{\int_0^a xy\,ax}{\text{area } ADC} = \frac{\int_0^a \frac{bx^2}{a}dx}{\frac{1}{2}ab} = \tfrac{2}{3}a;$$

that is, the distance of the centre of gravity from the vertex of the triangle is equal to *two thirds of the altitude of the triangle.*

2. Determine the centre of gravity of the area bounded by the parabola $y^2 = 4ax$ and the double ordinate ($2y$) perpendicular to the axis of x. $x' = \frac{3}{5}x,\ y' = 0.$

3. Find the centre of gravity of the area of the curve $xy^2 = b^2(a - x)$. $x' = \frac{1}{4}a,\ y' = 0.$

4. Find the centre of gravity of the area of the cycloid.
$x' = \pi r,\ y' = \frac{5}{6}r.$

5. Find the centre of gravity of the area of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ lying in the first quadrant. $x' = y' = \frac{256}{315}\frac{a}{\pi}.$

6. Find the centre of gravity of the area of $a^2y^2 + b^2x^2 = a^2b^2$, lying in the first quadrant. $x' = \frac{4a}{3\pi},\ y' = \frac{4b}{3\pi}.$

7. Find the centre of gravity of the arc of the circle $x^2 + y^2 = r^2$ lying in the first quadrant. (See formulas (R) and (S).)
$x' = y' = \frac{2r}{\pi}.$

8. Find the centre of gravity of the arc of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ lying in the first quadrant. $x' = y' = \frac{2}{5}a.$

9. Find the centre of gravity of the arc of the cycloid

$$x = r \operatorname{vers}^{-1} \frac{y}{r} - \sqrt{2ry - y^2},$$

lying between (0, 0) and $(\pi r, 2r)$. $x' = \frac{4}{3}r,\ y' = +\frac{4}{3}r.$

10. Find the centre of gravity of the paraboloid generated by revolving $y^2 = 4mx$ about the axis of x. (See formula (T).)

$x' = \frac{2}{3}x.$

11. Find the centre of gravity of the segment of a sphere generated by revolving $y^2 = 2rx - x^2$ about the axis of x.

$$x' = \frac{x(8r - 3x)}{4(3r - x)}.$$

When $x = r$, $x' = \frac{5}{8}r.$

12. A semi-ellipsoid is formed by the revolution of a semi-ellipse about its major axis; find the distance of the centre of gravity of the solid from the centre of the ellipse. $x' = \frac{3}{8}a.$

13. Find the centre of gravity of the convex surface of the cone generated by revolving the line $y = mx$ about the axis of x. (See formula (U).) $x' = \frac{2}{3}x.$

14. Find the centre of gravity of the surface of a spherical segment whose altitude is x. $x' = \frac{1}{2}x.$

15. Find the centre of gravity of the surface of the paraboloid generated by revolving $y^2 = 4mx$ about the axis of x.

$$x' = \frac{1}{5}\,\frac{(3x - 2m)(x + m)^{\frac{3}{2}} + 2m^{\frac{5}{2}}}{(x + m)^{\frac{3}{2}} - m^{\frac{3}{2}}}.$$

APPENDIX.

A_1. **Differentiable Functions.** A function, $y = f(x)$, is said to be differentiable when $\frac{\Delta y}{\Delta x}$ approaches a definite limit as Δx approaches zero. Thus, $y = \sqrt{x}$ is differentiable, since (Art. 10, ex. 5)

$$\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x' + h} + \sqrt{x'}}$$

approaches the definite limit, $\frac{1}{2\sqrt{x'}}$, as h approaches zero, x' being any particular or definite value of x from which h is estimated.

All ordinary continuous functions are differentiable, but this does not follow from the mere fact that the functions are continuous, for there are functions which are continuous and yet have no differential coefficients.* Functions of this kind, however, are of such rare occurrence that the distinction between continuity and differentiability is seldom made in works on the Differential Calculus. That is, every function is regarded as continuous and differentiable between certain limits.

The limit (m_1) of $\frac{\Delta y}{\Delta x}$ in any particular case can often be conveniently determined by assuming that $\Delta y = m_1 h + m_2 h^2$,

---

* See Harkness and Morley's Theory of Functions.

which is true of all differentiable functions of a single variable, and then finding the value of m_1, as in A_4. The general values of m_1 and m_2, and the exact conditions under which $\Delta y = m_1h + m_2h^2$ holds, are given in A_7.

A_2. **Another Illustration of the Formula** $\Delta y = m_1h + m_2h^2$. Suppose that a moving body has traversed a distance (s) in the time t, and that the value of s in terms of t is

$$s = f(t). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Suppose we wish to find the actual velocity (v_1) at the end of the time t_1. Let Δt, an increment of t estimated from t_1, be any arbitrary period immediately succeeding the end of the time t_1, then the distance traversed by the body in that period will be the corresponding increment of s, viz.,

$$\Delta s = f(t + \Delta t) - f(t) = m_1\Delta t + m_2(\Delta t)^2. \quad . \quad . \quad (2)$$

The mean velocity (v) of a moving body, during any period of time, is the quotient obtained by dividing the distance traversed by the body by the length of the period. Therefore the mean velocity during the period Δt is

$$\frac{\Delta s}{\Delta t} = m_1 + m_2\Delta t = v. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now this mean velocity evidently approaches the actual velocity v_1 as Δt approaches 0, indefinitely. Hence taking the limit of (3), we have

$$\text{limit of } \frac{\Delta s}{\Delta t} = m_1 = v_1.$$

That is, the limit of $\frac{\Delta s}{\Delta t}$, or m_1, is the actual velocity of the body at the end of the time t_1, and hence $m_1\Delta t$ is what Δs would have been had it varied as Δt or had the actual velocity v_1 remained constant, and m_2h^2 is the acceleration of s during the period Δt.

A_3. **The Differentials of Independent Variables are, in general, Variables.** In differentiating $y = f(x)$ successively dx is usually regarded as a constant; that is, as having the same value for all values of x. "This hypothesis," say Rice and Johnson,* "greatly simplifies the expressions for the second and higher differentials of functions of x, inasmuch as it is evidently equivalent to making all differentials of x higher than the first vanish." Again, "A differential of the second order or of a higher order," says Byerly,† "has been defined by the aid of a derivative, which always implies the distinction between function and variable, and on the hypothesis of an important difference in the natures of the increments of function and variable; namely, that the increment of the independent variable is a constant magnitude, and that, consequently, its derivative and differential are zero."

The impressions which these and similar statements in other excellent works are likely to make on the mind of the student are (*a*) that all independent variables vary uniformly, and (*b*) that they must vary in this manner in order that the differentials of their differentials shall be zero.

That an independent variable may vary uniformly, as in Rate of Change, is granted; but differentials in general are variables whose limits are zero. Indeed one of the most important and essential properties of the differential of an independent variable is its *independent variability.* The imposition of any condition on a group of variables by which they may be expressed in terms of one another at once destroys the independence of the variables, and this is the case of variables under the hypothesis of uniform variation, or rate of change.

Thus, let $u = f(x, y)$, and let us suppose x and y to vary simultaneously and uniformly; then $dx = mdt$ and $dy = m'dt$; whence $x = mt + C$ and $y = m't + C'$. Eliminating t and solving for y, we have $y = \phi(x)$. Hence the supposition of uniform change renders the hypothesis of more than one in-

---

* Dif. Calculus, Art. 79. † Dif. Calculus, Art. 204.

dependent variable impossible. Therefore, if independent variables (which are supposed to vary simultaneously) must vary uniformly in order that their higher differentials shall be zero, the successive differentials of $u = f(y, x, z)$ can be obtained only by destroying the independence of all the variables x, y, z except one.

Hence, in general, if the differential of dx, with respect to x, is zero, it is due to the fact that dx is a variable which is independent of x.

A_4. **To differentiate a^v and $\log_a v$ independently of Art. 78.**

Let $y = a^v$, where v is a function of x.

Increasing x by h, etc., and assuming that a^v is differentiable (A_1), we have

$$\frac{\Delta y}{\Delta v} = a^v\left(\frac{a^{\Delta v} - 1}{\Delta v}\right) = m_1 + m_2 \Delta v.$$

Taking the limits, remembering that $m_2 \Delta v$ vanishes with h, and that a^v is constant with respect to h, we have

$$\frac{dy}{dv} = a^v\left(\underset{\Delta v \doteq 0}{\text{limit}}\left[\frac{a^{\Delta v} - 1}{\Delta v}\right]\right) = m_1,$$

where m_1 and a^v are definite quantities. Therefore the limit of $\dfrac{a^{\Delta v} - 1}{\Delta v}$ must be a definite quantity (m' say), not zero, and dependent solely on the base a, since it is evidently independent of x and h.

$$\therefore \qquad \frac{dy}{dv} = a^v m', \quad \text{or} \quad dy = a^v m' dv. \quad . \quad . \quad . \quad . \quad (1)$$

Cor. I. Since m' depends on the base and the base is arbitrary, we may suppose the base to have such a value (say e) that $m' = 1$. Then $de^v = e^v dv$.

Cor. II. To find the value of m' in (1), let

$$a^v = e^u. \quad \quad (2)$$

Differentiating (2), $\quad a^v m' dv = e^u du. \quad \quad (3)$

From (2) and (3), $\quad m' = \dfrac{du}{dv}. \quad \quad (4)$

Taking $\log_e$ of (2), $\quad v \log_e a = u. \quad \quad (5)$

Differentiating (5), $\quad dv \log_e a = du. \quad \quad (6)$

Whence $\quad \log_e a = \dfrac{du}{dv}. \quad \quad (7)$

Equating (4) and (7), $\quad m' = \log_e a. \quad \quad (8)$

$$\therefore \quad d(a^v) = a^v \log_e a\, dv. \quad \quad (9)$$

Cor. III. To differentiate $y = \log_a v$, we write it under the form of

$$a^y = v. \quad \quad (10)$$

Differentiating (10), $\quad a^y \log_e a dy = dv;$

whence $\quad dy = \dfrac{1}{\log_e a} \dfrac{dv}{v} \quad \text{or} \quad \log_a e \dfrac{dv}{v}. \quad \quad (11)$

A_5. **A Rigorous Proof of Taylor's Formula.** In what follows, the function $f(y)$ and its n successive derivatives are supposed to be differentiable, and finite and continuous between the limits y and $y + x$.

Lemma. If $F(z)$ is continuous betwen $z = a$ and $z = y$, and if $F(a) = F(y) = 0$, then $F'(z)$, if continuous, must equal zero for some value of z between a and y.

For, as z changes from a to y, $F(z)$ passes from 0 to 0, that is, $F(z)$ increases and then decreases, or *vice versa;* hence $F'(z)$ must change from $+$ to $-$ or from $-$ to $+$, and therefore, since it is continuous, pass through 0.

In what follows θ will represent a positive proper fraction; that is, $0 < \theta < 1$. Hence, $0 < \theta x < x$, and, by the lemma, $F'[y + \theta(a - y)] = 0$.

Under the given hypotheses, we have (A_1)

$$f^{n-1}(y + x) = f^{n-1}(y) + m_1 x + m_2 x^2,$$

or

$$f^{n-1}(y + x) = f^{n-1}(y) + xs, \quad \ldots \ldots \quad (1)$$

where s $(= m_1 + m_2 x)$ is continuous between y and $y + x$.

Multiply (1) by dx, regard y constant, and integrate, and we have

$$f^{n-2}(y + x) = f^{n-2}(y) + xf^{n-1}(y) + \int xs\, dx, \quad . \; . \quad (2)$$

the constant C being $f^{n-2}(y)$, since $f^{n-2}(y + x) = f^{n-2}(y)$ when $x = 0$.

Multiply (2) by dx, and integrate, denoting $\int\int$ by $\int^2$, and we have

$$f^{n-3}(y + x) = f^{n-3}(y) + xf^{n-2}(y) + \frac{x^2}{\underline{|2}} f^{n-1}(y) + \int^2 xs\, dx^2, \quad (3)$$

C being $f^{n-3}(y)$.

Continuing thus to $n - 1$ integrations, we have

$$f(y + x) = f(y) + xf'(y) + \frac{x^2}{\underline{|2}} f''(y)$$
$$+ \ldots \frac{x^{n-1}}{\underline{|n - 1}} f^{n-1}(y) + \int^{n-1} xs\, dx^{n-1}. \quad (4)$$

We wish now to find the value of the last term, $\int^{n-1} xs\, dx^{n-1}$, which is the remainder after n terms.

In (4) put $a - y$ for x, and s' for the corresponding value of s, transpose, and we have

$$f(a) - f(y) - \frac{a - y}{1} f'(y) - \frac{(a - y)^2}{\underline{|2}} f''(y)$$
$$- \ldots \frac{(a - y)^{n-1}}{\underline{|n - 1}} f^{n-1}(y) - \int^{n-1} (a - y)s\, (-dy)^{n-1} = 0. \quad (5)$$

Let $F(z)$ be a function such that

$$F(z) = f(a) - f(z) - \frac{a-z}{1}f'(z) - \frac{(a-z)^2}{\underline{|2}}f''z$$

$$- \ldots \frac{(a-z)^{n-1}}{n-1}f^{n-1}(z) - \int^{n-1}(a-z)s'(-dz)^{n-1} \ldots \quad (6)$$

Since s', having the same value as in (5), is independent of z, the last term of (6) is equal to $\frac{(a-z)^n}{\underline{|n}}s'$.

Evidently $F(z) = 0$, *first* when $z = y$ by (5), *second* when $z = a$; and since $f(y), f'(y), f''(y)$, etc., are all continuous from y to $y + x$ $(= a)$, $f(z), f'(z), f''(z)$, etc. (and therefore $F(z)$ and $F'(z)$), are continuous between the same limits. Therefore, by the lemma, $F'(y + \theta(a - y)) = 0$.

Differentiating (6) to obtain $F'(z)$, we have

$$F'(z) = -f'(z) + f'(z) - \frac{(a-z)}{1}f''(z) + \frac{a-z}{1}f''(z)$$

$$\frac{(a-z)^2}{2}f'''(z) + \frac{(a-z)^2}{2}f'''(z) \ldots$$

$$+ \frac{(a-z)^{n-1}}{\underline{|n-1}}f^n(z) + \frac{(a-z)^{n-1}}{\underline{|n-1}}s'.$$

That is, $$F'(z) = -\frac{(a-z)^{n-1}}{\underline{|n-1}}[f^n(z) - s']. \quad . \quad . \quad . \quad (7)$$

Now substituting $y + \theta(a - y)$ for z, observing that for this value of z, $F'(z) = 0$, and dividing by $\frac{(a-z)^{n-1}}{\underline{|n-1}}$, we have

$$s' = f^n[y + \theta(a - y)] \ldots \quad . \quad . \quad . \quad . \quad . \quad (8)$$

In (6) substitute this value of s', and then put x for z

(whence $F(z) = 0$), $y + x$ for a (whence $a - y = x$), transpose, and we have

$$f(y + x) = f(y) + xf'(y) + \frac{x^2}{\lfloor 2}f''(y)$$

$$+ \dots \frac{x^{n-1}}{\lfloor n-1}f^{n-1}(y) + \frac{x^n}{\lfloor n}f^n(y + \theta x). \quad (9)$$

The last term

$$\frac{x^n}{\lfloor n}f^n(y + \theta x) = R_n, \text{ say,}$$

is called the remainder in Taylor's formula. It is obtained by differentiating $f(y)$ n times and substituting $y + \theta x$ for y in the final or nth derivative.

When the function $f(y + x)$ is such that this remainder approaches 0 as n approaches ∞, the series will be convergent, otherwise it will be divergent. Hence the remainder enables us to ascertain the conditions under which any given function of the sum of two variables is developable by Taylor's formula, and also to find the limits of the error we make in stopping at any term of the series.

EXAMPLE.

1. $f(y + x) = \log (y + x)$. See Art. 125.

Since $\quad f(y) = \log y, \quad f^n(y) = -(-1)^n \dfrac{\lfloor n-1}{y^n}.$

$$\therefore \qquad R_n = -(-1)^n \frac{1}{n}\left(\frac{x}{y + \theta x}\right)^n.$$

Now, since $0 < \theta < 1$, $\dfrac{x}{y + \theta x}$ is finite and a proper fraction if $x =$ or $< y$. Therefore R_n approaches 0 as n approaches ∞, and $\log (y + x)$ is developable if $x =$ or $< y$.

Again, since $0 < \theta < 1$, the true numerical value of R_n lies between $\frac{1}{n}\left(\frac{x}{y}\right)^n$ and $\frac{1}{n}\left(\frac{x}{y+x}\right)^n$. Hence if $-(-1)^n\frac{1}{n}\left(\frac{x}{y}\right)^n$ be substituted for R_n in (1), Art. 125, the series will be the value of $\log(y+x)$ to within less than $\frac{1}{n}\left(\frac{x}{y+x}\right)^n$.

A$_6$. **A Similar Completion of Maclaurin's Formula** may be obtained by making $y = 0$ in (9), A$_5$. Thus,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\underline{|2}}f''(0)$$

$$+ \dots \frac{x^{n-1}}{\underline{|n-1}}f^{n-1}(0) + \frac{x^n}{\underline{|n}}f^n(\theta x). \quad (1)$$

EXAMPLE.

$f(x) = e^x$.

Since $f^n(x) = e^x$, $\qquad R_n = \frac{x^n}{\underline{|n}}e^{\theta x}$.

For any finite value of x, (1) the fraction $\frac{x^n}{\underline{|n}}$ evidently approaches 0 as n approaches ∞, and (2), since $0 < \theta < 1$, $e^{\theta x}$ is finite. Hence the limit of R_n, as n approaches ∞, is 0, and e^x is developable, for all finite values of x.

Again, as $0 < \theta < 1$, the true value of R_n lies between $\frac{x^n}{\underline{|n}}$ and $\frac{x^n}{\underline{|n}}e^x$. Therefore the sum of all the terms after the nth in series (N), Art. 127, is less than $\frac{x^n}{\underline{|n}}e^x$.

A$_7$. **To determine the values of m_1 and m_2 in the formula $\Delta y = m_1h + m_2h^2$ (Art. 24).**

Since $y = f(x)$, $\qquad \Delta y = f(x+h) - f(x)$.

By Taylor's formula,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x + \theta h).$$

$$\therefore \qquad \Delta y = f'(x)h + \tfrac{1}{2}f''(x + \theta h)h^2.$$

Comparing this with the given formula, we have $m_1 = f'(x)$ and $m_2 = \frac{1}{2}f''(x + \theta h)$, where $0 < \theta < 1$.

Therefore the formula $\Delta y = m_1 h + m_2 h^2$ is true in reference to $y = f(x)$, (1) if $f(x)$ and $f'(x)$ are differentiable, and (2) if $f(x)$, $f'(x)$, and $f''(x)$ are finite and continuous between the limits x and $x + h$.